Reduction Groups and Automorphic Lie Algebras.

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Abstract

We study a new class of infinite dimensional Lie algebras, which has important applications to the theory of integrable equations. The construction of these algebras is very similar to the one for automorphic functions and this motivates the name *automorphic Lie algebras*. For automorphic Lie algebras we present bases in which they are quasigraded and all structure constants can be written out explicitly. These algebras have a useful factorisations on two subalgebras similar to the factorisation of the current algebra on the positive and negative parts.

1 Introduction

In this paper we introduce and study automorphic Lie algebras. This subclass of infinite dimensional Lie algebras is very useful for applications and actually has been motivated by applications to the theory of integrable equations. Automorphic Lie algebras are quasigraded and all their structure constants can be found explicitly. They form a more general class than graded infinite dimensional Lie algebras [1], they also have rich internal structure and can be studied in depth.

The basic construction is very similar to the theory of automorphic functions [2], [3]. In a sense, it is a generalisation of this theory to the case of semi–simple Lie algebras over a ring of meromorphic functions $\mathcal{R}(\Gamma)$ of a complex parameter λ with poles in a set of points Γ. Suppose $\mathcal{G}$ is a discontinuous group of fractional-linear transformations of the complex variable λ and the set Γ is an orbit of this group or a finite union of orbits, then transformations from $\mathcal{G}$ induce automorphisms of the ring $\mathcal{R}(\Gamma)$. A set of elements of $\mathcal{R}(\Gamma)$ which are invariant with respect to $\mathcal{G}$ form a subring of automorphic functions. Automorphic algebras are defined in a very similar way. Let us consider a finite dimensional semi-simple Lie algebra $\mathcal{A}$ over the ring $\mathcal{R}(\Gamma)$. This algebra can be viewed as an infinite dimensional Lie algebra over $\mathbb{C}$ and will be denoted $\mathcal{A}(\Gamma)$. Suppose G is a subgroup of the group of automorphisms of $\mathcal{A}(\Gamma)$. Elements of G are simultaneous transformations (automorphisms) of the semi-simple Lie algebra $\mathcal{A}$ and the ring $\mathcal{R}(\Gamma)$. Then the automorphic Lie algebra $\mathcal{A}_G(\Gamma)$ is defined as the set of all elements of $\mathcal{A}(\Gamma)$ which are G invariant.

In this paper we restrict ourselves to finite groups of fractional-linear transformations of the Riemann sphere and therefore the set Γ is finite and all elements of $\mathcal{R}(\Gamma)$ are rational functions. The theory of automorphic functions for finite groups has been developed by Felix Klein [2], [4]; automorphic functions corresponding to finite groups can be easily obtained using the group average. The paper is organised as follows: in the second section we introduce notations and recall some useful results from the theory of elementary automorphic functions. We give a brief account of automorphisms of semi-simple Lie algebras, discuss the structure of automorphisms groups of algebras over a ring of rational functions and define automorphic Lie algebras. In the third section we construct explicitly automorphic Lie algebras

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corresponding to the dihedral group $\mathbb{D}_N$ and study some of their properties. In particular we built explicitly bases in which these algebras are quasigraded and find all structure constants. The group of automorphisms of a semi-simple Lie algebra is a continuous Lie group and therefore its elements may depend on the complex parameter λ. In this case the reduction group G is a subgroup of a semi-direct product of $\mathcal{G}$ and Aut $\mathcal{A}$. A nontrivial example of the corresponding automorphic Lie algebra is given in section 3.3. For completeness, in the Appendix we give an account of all finite groups of fractional linear transformations, their orbits and primitive automorphic functions.

Originally our study has been motivated by the problem of reduction of Lax pairs. Most of integrable equations interesting for applications are results of reductions of bigger systems. The problem of reductions is one of the central problems in the theory of integrable equations. A wide class of algebraic reductions can be studied in terms of *reduction groups*. The concept of reduction group has been formulated in [5], [6], [7] and developed in [8], [9], [10], [11]. It has been successfully applied and proved to be very useful for a classification of solutions of the classical Yang-Baxter equation [12], [13]. The most recent publications related to the reduction group are [14], [15].

A reduction group G is a discrete group of automorphisms of a Lax pair. Its elements are simultaneous gauge transformations and fractional-linear transformations of the spectral parameter. The requirement that a Lax pair is invariant with respect to a reduction group imposes certain constraints on the entries of the Lax pair and yields a reduction. Simultaneous gauge transformations and fractional-linear transformations of the spectral parameter are automorphisms of the underlying infinite dimensional Lie algebra $\mathcal{A}(\Gamma)$. The reduction corresponding to G is nothing but a restriction of the Lax pair to the automorphic subalgebra $\mathcal{A}_G(\Gamma) \subset \mathcal{A}(\Gamma)$.

About a year ago we discussed our new developments in the theory of reductions and reduction groups [14] with V.V. Sokolov, who suggested us to reformulate our results in algebraic terms in order to make them accessible to a wider mathematical community. We are grateful to him for this advise. Indeed, Lie algebras have applications far beyond the theory of integrable equations. We believe *automorphic Lie algebras* are a new and important class of infinite dimensional Lie algebras which deserves further study and development.

2 Automorphisms

2.1 Finite groups of automorphisms of the complex plane and rational automorphic functions

Let $\hat{\mathcal{G}}$ be a group of fractional-linear transformations σ_r

$$\lambda_r = \sigma_r(\lambda) = \frac{a_r \lambda + b_r}{c_r \lambda + d_r}, \quad a_r d_r - b_r c_r = 1, \tag{1}$$

where σ_0 is the identity transformation (*id*) of the group

$$\sigma_0(\lambda) = \lambda, \quad a_0 = d_0 = \pm 1, \quad b_0 = c_0 = 0.$$

The composition $\sigma_{r'}(\sigma_r(\lambda))$ defines the group product $\sigma_{r'} \cdot \sigma_r$. We will denote $\sigma_r^{-1}(\lambda)$ the transformation inverse to $\sigma_r(\lambda)$. One can associate 2×2 matrices with fractional-linear transformations (1)

$$\begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} \rightarrow \sigma_r.$$

The product of such matrices corresponds to the composition of fractional-linear transformations. It defines a homomorphism of the group $SL(2, \mathbb{C})$ onto the group $\hat{\mathcal{G}}$. The kernel of the homomorphism consists of two elements I_2 and $-I_2$ where I_2 is the unit 2×2 matrix. In other words, the group $\hat{\mathcal{G}}$ is isomorphic to $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I_2\}$.

Two groups $\mathcal{G}$ and $\mathcal{G}'$ of fractional-linear transformations are equivalent if there is a fractional-linear transformation τ such that for any $\sigma \in \mathcal{G}$

$$\sigma' = \tau^{-1}\sigma\tau \in \mathcal{G}'$$

and any element of $\mathcal{G}'$ can be obtained in this way.

Finite subgroups of $\hat{\mathcal{G}}$ have been completely classified by Felix Klein [4]. The complete list of finite groups of fractional-linear transformations consists of five elements:

$$\mathbb{Z}_N, \quad \mathbb{D}_N, \quad \mathbb{T}, \quad \mathbb{O}, \quad \mathbb{I}, \tag{2}$$

i.e. the additive group of integers modulo N, the group of a dihedron with N corners, the tetrahedral, octahedral and icosahedral groups, respectively. In this paper we consider only finite groups of fractional-linear transformations.

Let γ_0 be a complex number (a point on the Riemann sphere $\mathbb{CP}^1$), and let $\mathcal{G}$ be a finite group of fractional-linear transformations. The *orbit* $\mathcal{G}(\gamma_0)$ is defined as the set of all images $\mathcal{G}(\gamma_0) = \{\sigma_r(\gamma_0) \,|\, \sigma_r \in \mathcal{G}\}$. If two orbits $\mathcal{G}(\gamma_1)$ and $\mathcal{G}(\gamma_2)$ have non-empty intersection, they coincide. The point γ_0 is called a *fixed point* of a transformation σ_r if $\sigma_r(\gamma_0) = \gamma_0$. Transformations for which the point γ_0 is fixed form a subgroup $\mathcal{G}_{\gamma_0} \subset \mathcal{G}$, called *isotropy subgroup* of γ_0. The *order* of the fixed point is defined as the order of its isotropy subgroup $\mathrm{ord}(\gamma_0) = |\mathcal{G}_{\gamma_0}|$. The point γ_0 and the corresponding orbit $\mathcal{G}(\gamma_0)$ are called *generic*, if the isotropy subgroup $\mathcal{G}_{\gamma_0}$ is trivial, i.e. it consists of the identity transformation only. The orbit $\mathcal{G}(\gamma_0)$, and so γ_0, is called *degenerated*, if $|\mathcal{G}_{\gamma_0}| > 1$. It follows from Lagrange Theorem that the number of points in the orbit $\mathcal{G}(\gamma_0)$ is equal to $|\mathcal{G}|/|\mathcal{G}_{\gamma_0}|$.

Given a rational function $f(\lambda)$ of the complex variable λ, the action of the group $\mathcal{G}$ is defined as

$$\sigma_r : f(\lambda) \rightarrow f(\sigma_r^{-1}(\lambda)), \tag{3}$$

or simply $\sigma_r(f(\lambda)) = f(\sigma_r^{-1}(\lambda))$. A non-constant function $f(\lambda)$ is called *automorphic function* of the group $\mathcal{G}$ if $\sigma_r(f(\lambda)) = f(\lambda)$ for all $\sigma_r \in \mathcal{G}$. Automorphic functions take the same value at all points of an orbit $\mathcal{G}(\gamma_0)$.

The following important fact holds; it has been perfectly known to Felix Klein, but it was not formulated as a separate statement in his book [2].

Theorem 2.1 *Let $\mathcal{G}$ be a finite group of fractional-linear transformations, and be $\mathcal{G}(\gamma_1), \mathcal{G}(\gamma_2)$ any two different orbits, then:*

1. *There exists a* primitive *automorphic function $f(\lambda, \gamma_1, \gamma_2)$ with poles of multiplicity $|\mathcal{G}_{\gamma_1}|$ at points $\mathcal{G}(\gamma_1)$ and zeros of multiplicity $|\mathcal{G}_{\gamma_2}|$ at points $\mathcal{G}(\gamma_2)$ and with no other poles or zeros. Function $f(\lambda, \gamma_1, \gamma_2)$ is defined uniquely, up to a constant multiplier.*

2. *Any rational automorphic function of the group $\mathcal{G}$ is a rational function of the* primitive *$f(\lambda, \gamma_1, \gamma_2)$.*

If $f(\lambda, \gamma_1, \gamma_2)$ is a primitive automorphic function, then

$$f(\lambda, \gamma_2, \gamma_1) = \frac{c_1}{f(\lambda, \gamma_1, \gamma_2)}, \tag{4}$$

$$f(\lambda, \gamma_1, \gamma_3) = c_2(f(\lambda, \gamma_1, \gamma_2) - f(\gamma_3, \gamma_1, \gamma_2)), \quad \gamma_3 \notin \mathcal{G}(\gamma_1), \tag{5}$$

$$f(\lambda, \gamma_3, \gamma_4) = c_3 \frac{f(\lambda, \gamma_1, \gamma_2) - f(\gamma_4, \gamma_1, \gamma_2)}{f(\lambda, \gamma_1, \gamma_2) - f(\gamma_3, \gamma_1, \gamma_2)}, \quad \gamma_3, \gamma_4 \notin \mathcal{G}(\gamma_1), \tag{6}$$

where c_1, c_2, c_3 are nonzero complex constants. Thus, it is sufficient to find one primitive automorphic function $f = f(\lambda, \gamma_1, \gamma_2)$ and all other rational automorphic functions will be rational functions of f.

For finite groups, automorphic functions can be obtained using the group average

$$\langle f(\lambda) \rangle = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \sigma(f(\lambda)) . \tag{7}$$

In order to obtain a primitive function $f(\lambda, \gamma_1, \gamma_2)$ we define the automorphic function

$$\hat{f}(\lambda, \gamma_1) = \langle \frac{1}{(\lambda - \gamma_1)^{|\mathcal{G}_{\gamma_1}|}} \rangle = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \frac{1}{(\sigma^{-1}(\lambda) - \gamma_1)^{|\mathcal{G}_{\gamma_1}|}} \tag{8}$$

with poles of multiplicity $|\mathcal{G}_{\gamma_1}|$ at points of the orbit $\mathcal{G}(\gamma_1)$ and then $f(\lambda, \gamma_1, \gamma_2) = \hat{f}(\lambda, \gamma_1) - \hat{f}(\gamma_2, \gamma_1)$. It is essential that the order of the pole in (8) has been chosen equal to the order of the fixed point γ_1. If the order is less than $|\mathcal{G}_{\gamma_1}|$, then the group average is a constant function, i.e. it does not depend on λ.

For completeness, in the Appendix we give an account of all finite groups (2) of fractional-linear transformations, their orbits and corresponding primitive automorphic functions.

2.2 Automorphisms of semi-simple Lie algebras

The structure of the automorphisms groups of semi-simple Lie algebras over $\mathbb{C}$ is comprehensively studied (see for example the book of Jacobson [16]). In this section we list some results which will be used in the rest of the text.

Let $\mathcal{A}$ be a finite or infinite dimensional Lie algebra over any field (or ring). We denote by $\operatorname{Aut} \mathcal{A}$ the group of all automorphisms of $\mathcal{A}$. Let $G \subset \operatorname{Aut} \mathcal{A}$ be a subgroup and $\mathcal{A}_G$ be a subset of all elements of $\mathcal{A}$ which are invariant with respect to all transformations of G, i.e.

$$\mathcal{A}_G = \{ a \in \mathcal{A} \,|\, \phi(a) = a \,, \forall \phi \in G \} .$$

Lemma 2.2 $\mathcal{A}_G$ *is a subalgebra of $\mathcal{A}$.*

This lemma is obvious (it follows immediately from the automorphism definition), but important for our further applications. All classical semi-simple Lie algebras can be extracted in such a way from the algebra of matrices with zero trace. For example, the map $\phi_t(a) = -a^{tr}$, where a^{tr} stands for the transpose matrix, is an automorphism of the Lie algebra $sl(N, \mathbb{C})$ of square $N \times N$ matrices. The invariant subalgebra in this case is $so(N, \mathbb{C})$, i.e. the algebra of skew-symmetric matrices.

¿From now on we assume that $\mathcal{A}$ is a finite dimensional semi-simple Lie algebra over $\mathbb{C}$. The group $\operatorname{Aut} \mathcal{A}$ is a Lie group. It is generated by inner automorphisms of the form $\phi_{in} = e^{ad_a}, a \in \mathcal{A}$ and outer automorphisms ϕ_{out}, induced by automorphisms (symmetries) of the Dynkin diagram of $\mathcal{A}$. Any automorphism $\phi \in \operatorname{Aut} \mathcal{A}$ can be uniquely represented as a composition $\phi_{in} \cdot \phi_{out}$. Inner automorphisms form a Lie subgroup $\operatorname{Aut}_0 \mathcal{A}$ of the group $\operatorname{Aut} \mathcal{A}$. The subgroup $\operatorname{Aut}_0 \mathcal{A}$ is normal and a connected component of the identity of the group of all automorphisms. The algebras $A_n \,, (n > 1) \,, D_n \,, (n > 4)$ and E_6 have subgroups of outer automorphisms of order two, the algebra D_4 has the group $\operatorname{Aut} \mathcal{A}/\operatorname{Aut}_0 \mathcal{A} \cong S_3$, i.e. the group of permutations of three elements, of order six and isomorphic to $\mathbb{D}_3$. Other semi-simple Lie algebras do not admit outer automorphisms.

The description of the group of automorphisms can be given in explicit form. For example in the case of the algebra $sl(N, \mathbb{C})$ we have [16]:

Theorem 2.3 *The group of automorphisms of the Lie algebra of 2×2 matrices of zero trace is a set of mappings $a \to QaQ^{-1}$. The group of automorphisms of the Lie algebra of $N \times N, N > 2$, matrices of trace 0 is a set of mappings $a \to QaQ^{-1}$ and $a \to -Ha^{tr}H^{-1}$. Where $Q, H \in GL(N, \mathbb{C})$.*

Explicit descriptions of the groups of automorphisms for other semi-simple algebras can be found in [16]. In this paper we focus on the study of Lie subalgebras related to $sl(N, \mathbb{C})$.

4

2.3 Automorphisms of Lie algebras over rings of rational functions. Automorphic Lie algebras

A straightforward application of Lemma 2.2 to finite dimensional semi-simple Lie algebras does not lead to interesting results. Indeed, if we wish the invariant subalgebra $\mathcal{A}_G$ to be semi-simple we are coming back to the famous list of the Cartan classification and nothing new can be found on this way. Infinite dimensional Lie algebra with elements depending on a complex parameter λ may have a richer group of automorphisms and Lemma 2.2 provides a tool to construct subalgebras of infinite dimensional Lie algebras in the spirit of the theory of automorphic functions [2], [3].

Let $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ be a finite set of points $\gamma_k \in \hat{\mathbb{C}} \simeq \mathbb{CP}^1$. The linear space of all rational functions of a complex variable $\lambda \in \mathbb{C}$ which may have poles of any finite order at points of Γ and no other singularities in $\hat{\mathbb{C}}$, equipped with usual multiplication, form a ring $\mathcal{R}(\Gamma)$ and $\mathbb{C} \subset \mathcal{R}(\Gamma)$. The ring $\mathcal{R}(\Gamma)$, as a linear space of functions over $\mathbb{C}$, is infinite dimensional. Let $\mathcal{A}$ be a finite dimensional semi–simple Lie algebra over $\mathbb{C}$. We define

$$\mathcal{A}(\Gamma) = \{\sum_k f_k(\lambda)\, e_k \mid f_k \in \mathcal{R}(\Gamma)\,,\ e_k \in \mathcal{A}\}\,, \tag{9}$$

with standard commutator

$$\left[\sum_k f_k(\lambda) e_k\,,\ \sum_s g_s(\lambda) e_s\right] = \sum_{k,s} f_k(\lambda) g_s(\lambda)\, [e_k\,,\ e_s]\,. \tag{10}$$

The algebra $\mathcal{A}(\Gamma)$ is an infinite dimensional Lie algebra over $\mathbb{C}$. The group of automorphisms of $\mathcal{A}(\Gamma)$, $\operatorname{Aut} \mathcal{A}(\Gamma)$, may be richer then $\operatorname{Aut} \mathcal{A}$; indeed, let Γ be an orbit or a finite union of orbits of a finite group $\mathcal{G}$ of fractional-linear transformations, then the ring $\mathcal{R}(\Gamma)$ has a nontrivial group of automorphisms $\operatorname{Aut} \mathcal{R}(\Gamma) \cong \mathcal{G}$. The group $\operatorname{Aut} \mathcal{R}(\Gamma)$ is the group of all automorphisms of the ring which do not move the base field of constants (i.e. $\mathbb{C}$). Automorphisms of the ring induce automorphisms of the algebra $\mathcal{A}(\Gamma)$. The direct product of the groups $\operatorname{Aut} \mathcal{R}(\Gamma) \times \operatorname{Aut} \mathcal{A}$ is a group of automorphisms of $\mathcal{A}(\Gamma)$. It can be generalised to a semi-direct product, if there is a nontrivial homomorphism of $\operatorname{Aut} \mathcal{R}(\Gamma)$ in the group $\operatorname{Aut} (\operatorname{Aut} \mathcal{A})$ (an example will be given in Section 3.3). In the rest of the article we assume that the set Γ is an orbit or a union of a finite number of orbits of a finite group $\mathcal{G}$ of fractional-linear transformations.

For any group H and two monomorphisms $\tau : H \to A$ and $\psi : H \to B$, the *diagonal* subgroup of the direct product $\tau(H) \times \psi(H)$ is defined as

$$\operatorname{diag}(\tau(H) \times \psi(H)) = \{(\tau(h), \psi(h)) \mid h \in H\}\,.$$

Let A and B be two groups and G be a subgroup of the direct product $G \subset A \times B$. Each element $g \in G$ is a pair $g = (\alpha, \beta)$, where $\alpha \in A$ and $\beta \in B$. There are two natural projections π_1, π_2 on the first and the second components of the pair

$$\pi_1(g) = \alpha\,, \qquad \pi_2(g) = \beta\,.$$

Theorem 2.4 *Let $G \subset A \times B$ be a subgroup of the direct product of two groups A, B, and let*

$$U_1 = G \cap (A \times id)\,, \qquad U_2 = G \cap (id \times B)\,, \qquad K = U_1 \cdot U_2\,.$$
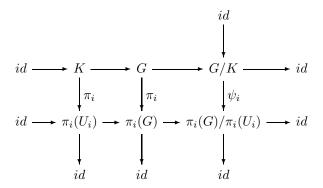
Then:

1. *U_1, U_2 and K are normal subgroups of G.*

2. *$\pi_i(U_i)$ is a normal subgroup of $\pi_i(G)$, $i = 1, 2$.*

3. There are two isomorphisms

$$\psi_1 : G/K \to \pi_1(G)/\pi_1(U_1)\,, \quad \psi_2 : G/K \to \pi_2(G)/\pi_2(U_2)\,.$$

4. $G/K \cong diag\ (\psi_1(G/K) \times \psi_2(G/K))\,.$

The proof of the theorem becomes obvious if we represent it in terms of two commutative diagrams $(i = 1,\, 2)$ with exact horizontal and vertical sequences of group homomorphisms:

Definition 2.5 *Let $G \subset Aut\,\mathcal{A}(\Gamma)$, we call the Lie algebra $\mathcal{A}_G(\Gamma)$ automorphic, if its elements $a \in \mathcal{A}_G(\Gamma)$ are invariant $g(a) = a$ with respect to all automorphisms $g \in G$. Group G is called the* reduction group.

The set $\mathcal{A}_G(\Gamma) = \{a \in \mathcal{A}(\Gamma) \,|\, g(a) = a,\ \forall g \in G\}$ is a subalgebra of $\mathcal{A}(\Gamma)$ (Lemma 2.2).

Like automorphic functions, automorphic subalgebras of $\mathcal{A}(\Gamma)$ can be constructed (in the case of a finite group G) using the group average. For any element $a \in \mathcal{A}(\Gamma)$ we define (compare with (7))

$$\langle a \rangle_G = \frac{1}{|G|} \sum_{g \in G} g(a)\,. \tag{11}$$

The group average is a liner operator in the linear space $\mathcal{A}(\Gamma)$ over $\mathbb{C}$, moreover, it is a projector, since $\langle \langle a \rangle_G \rangle_G = \langle a \rangle_G$ for any element $a \in \mathcal{A}(\Gamma)$.

If the group G has a normal subgroup $N \subset G$ then we can perform the average in two stages: first we take the average over the normal subgroup $\bar{a} = \langle a \rangle_N$ and then take the average over the factor group $\langle \bar{a} \rangle_{G/N}$

$$\langle a \rangle_G = \langle \langle a \rangle_N \rangle_{G/N}\,.$$

Let $[g]$ be a co-set in G/N and $\hat{g} \in [g]$ be one representative from the co-set, then the average $\langle \bar{a} \rangle_{G/N}$ is defined as

$$\langle \bar{a} \rangle_{G/N} = \frac{|N|}{|G|} \sum_{[\hat{g}] \in G/N} \hat{g}(\bar{a})\,.$$

This definition is well posed since $\hat{g}(\bar{a})$ is constant on each co-set $[g]$, i.e. the result does not depend on the choice of a representative.

If $G \subset \mathrm{Aut}\,\mathcal{R}(\Gamma) \times \mathrm{Aut}\,\mathcal{A}$, and it has nontrivial normal subgroups U_1, U_2 (in the notation of Theorem 2.4) then

$$\mathcal{A}_G(\Gamma) = \langle \mathcal{A}(\Gamma) \rangle_G = \langle \langle \langle \mathcal{A}(\Gamma) \rangle_{U_1} \rangle_{U_2} \rangle_{G/K} = \langle \langle \langle \mathcal{A}(\Gamma) \rangle_{U_2} \rangle_{U_1} \rangle_{G/K}\,. \tag{12}$$

The normal subgroup U_1 of a reduction group G consists of all elements of the form (σ, id), it corresponds to fractional-linear transformations of the complex variable λ, and identical transformation of the

algebra $\mathcal{A}$. The normal subgroup U_2 consists of all elements of the form (id, ϕ), i.e. automorphisms of $\mathcal{A}$ and identical transformation of the variable λ. The factor group G/K, if it is nontrivial, corresponds to simultaneous automorphisms of the ring $\mathcal{R}(\Gamma)$ and the algebra $\mathcal{A}$.

Averaging $\mathcal{A}(\Gamma)$ over U_2 is equivalent to a replacement of the algebra $\mathcal{A}$ by $\mathcal{A}_{\pi_2(U_2)}$ (Lemma 2.2). Thus, without any loss of generality, we can start from a smaller algebra $\mathcal{A}_{\pi_2(U_2)}$ and respectively a smaller reduction group $\hat{G} \cong G/U_2$.

Averaging over U_1 affects only the ring $\mathcal{R}(\Gamma)$. As the result, we receive a subring $\mathcal{R}_{\pi_1(U_1)}(\Gamma) \subset \mathcal{R}(\Gamma)$ of $\pi_1(U_1)$-automorphic functions with poles at Γ. It follows from the Theorem 2.1 that any element of $\mathcal{R}_{\pi_1(U_1)}(\Gamma)$ can be expressed as a rational function of a primitive $\pi_1(U_1)$-automorphic function. Taking a primitive automorphic function instead of λ, we reduce then the problem to a simpler one (with a trivial subgroup U_1), without any loss of generality.

Thus, the most interesting case corresponds to simultaneous transformations and from the very beginning we can assume the subgroup $K = U_1 \cdot U_2$ to be trivial, without any loss of generality. If K is trivial then $G \cong \mathcal{G} = \pi_1(G) \simeq \pi_2(G)$ (Theorem 2.1). If G is finite, it should be isomorphic to one of the finite groups of fractional-linear transformations (2). Thus, the reduction group $G = \text{diag}\,(\mathcal{G}, \psi(\mathcal{G}))$ where $\psi : \mathcal{G} \to \text{Aut}\,\mathcal{A}$ is a monomorphism of a finite group of fractional-linear transformations $\mathcal{G}$ into the group of automorphisms of Lie algebra $\mathcal{A}$.

The above construction can be generalised to the case in which the elements of $\text{Aut}\,\mathcal{A}$ are λ dependent. In this case, the composition law for the elements of the reduction group is similar to the one for a semi-direct product of groups. A nontrivial example of such generalisation and the corresponding automorphic Lie algebra will be discussed in section 3.3.

2.4   Quasigraded structure

Following I. M. Krichever and S. P. Novikov [17] we define a *quasigraded structure* for infinite dimensional Lie algebras.

Definition 2.6 *An infinite dimensional Lie algebra $\mathcal{L}$ is called* quasigraded, *if it admits a decomposition as a vector space in a direct sum of subspaces*

$$\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^n \tag{13}$$

and there exist two non negative integer constants p and q such that

$$[\mathcal{L}^n, \mathcal{L}^m] \subseteq \bigoplus_{-q \leq k \leq p} \mathcal{L}^{n+m+k} \quad \forall\, n, m \in \mathbb{Z}\,. \tag{14}$$

For $p = q = 0$ the algebra $\mathcal{L}$ is graded. Elements of $\mathcal{L}^n$ are called homogeneous elements of degree n. The decomposition (13) with the property (14) is called a quasigraded structure *of $\mathcal{L}$.*

Without loss of generality we can assume $q = 0$. Indeed, by a simple shift in the enumeration we can always set $q = 0$. Quasigraded algebras with $p = 1, q = 0$ share one important property with graded algebras ($p = q = 0$), namely they can be decomposed (splitted) into a sum of two subalgebras

$$\mathcal{L} = \mathcal{L}_+ \bigoplus \mathcal{L}_-$$

where

$$\mathcal{L}_+ = \bigoplus_{n \geq 0} \mathcal{L}^n\,, \quad \mathcal{L}_- = \bigoplus_{n < 0} \mathcal{L}^n\,.$$

Indeed, it follows from (14) that commutators of elements from subspaces with negative indexes belong to $\mathcal{L}_-$ since $n + m + 1 < 0$ if $n < 0$ and $m < 0$. Commutators of elements from $\mathcal{L}_+$ obviously belong to $\mathcal{L}_+$. If $q = 0$ and $p > 1$ then $\mathcal{L}_-$ is not necessarily a closed subalgebra, but $\mathcal{L}_+$ is.

3 Explicit construction of automorphic Lie algebras

To construct an automorphic Lie algebra we consider the following:

1. a finite group of fractional-linear transformations $\mathcal{G}$,

2. a finite dimensional semi-simple Lie algebra $\mathcal{A}$ over $\mathbb{C}$,

3. a monomorphism $\psi : \mathcal{G} \to \operatorname{Aut} \mathcal{A}$.

For a given $\mathcal{G}, \mathcal{A}$ and ψ, automorphic Lie algebras depend on the choice of a $\mathcal{G}$-invariant set Γ, which is a union of a finite number of orbits $\Gamma = \cup_{k=1}^{M} \mathcal{G}(\gamma_k)$. Similar to the theory of automorphic functions (Theorem 2.1), for any two orbits $\mathcal{G}(\gamma_1), \mathcal{G}(\gamma_2)$, there is a uniquely defined *primitive* automorphic Lie algebra $\mathcal{A}_{\mathcal{G}}(\gamma_1, \gamma_2)$, whose elements may have poles at points in $\mathcal{G}(\gamma_1) \cup \mathcal{G}(\gamma_2)$ and do not have any other singularities. Algebra $\mathcal{A}_{\mathcal{G}}(\gamma_1, \gamma_2)$ is *quasigraded* (see Definition 2.6) and its structure constants can be written explicitly. Structure constants of any other $\mathcal{G}$-automorphic Lie algebra can be explicitly expressed in terms of the structure constants of $\mathcal{A}_{\mathcal{G}}(\gamma_1, \gamma_2)$. In general, algebra $\mathcal{A}_{\mathcal{G}}(\gamma_1, \gamma_2)$ can be decomposed in a direct sum of three linear spaces

$$\mathcal{A}_{\mathcal{G}}(\gamma_1, \gamma_2) = \mathcal{A}_{\mathcal{G}}(\gamma_1) \bigoplus \mathcal{A}_{\mathcal{G}}^0 \bigoplus \mathcal{A}_{\mathcal{G}}(\gamma_2), \tag{15}$$

such that elements of $\mathcal{A}_{\mathcal{G}}(\gamma)$ may have poles at the points of the orbit $\mathcal{G}(\gamma)$ and are regular elsewhere and elements of a finite dimensional linear space $\mathcal{A}_{\mathcal{G}}^0$ are constants, i.e. they do not depend on λ. Often the subspace $\mathcal{A}_{\mathcal{G}}^0$ is empty, then $\mathcal{A}_{\mathcal{G}}(\gamma_1)$ and $\mathcal{A}_{\mathcal{G}}(\gamma_2)$ are subalgebras. In all cases studied we have found a subalgebra $\hat{\mathcal{A}}_{\mathcal{G}}(\gamma_1, \gamma_2) \subseteq \mathcal{A}_{\mathcal{G}}(\gamma_1, \gamma_2)$ which can be decomposed as a linear space in a direct sum

$$\hat{\mathcal{A}}_{\mathcal{G}}(\gamma_1, \gamma_2) = \hat{\mathcal{A}}_{\mathcal{G}}(\gamma_1) \bigoplus \hat{\mathcal{A}}_{\mathcal{G}}(\gamma_2), \tag{16}$$

such that $\hat{\mathcal{A}}_{\mathcal{G}}(\gamma_1)$ and $\hat{\mathcal{A}}_{\mathcal{G}}(\gamma_2)$ are subalgebras whose elements may have poles at the orbits $\mathcal{G}(\gamma_1)$ and $\mathcal{G}(\gamma_2)$ respectively and are regular elsewhere.

3.1 Simple example $\mathcal{G} = \mathbb{D}_N$, $\mathcal{A} = sl(2, \mathbb{C})$.

The action of the dihedral group $\mathbb{D}_N$ on the complex plane can be generated by two transformations $\sigma_s(\lambda) = \Omega \lambda$, with $\Omega = \exp(2i\pi/N)$ and $\sigma_t(\lambda) = \lambda^{-1}$ (see details in the Appendix). It follows from Theorem 2.3 that all automorphisms $\operatorname{Aut} sl(2, \mathbb{C})$ are inner and can be represented in the form $\phi(a) = QaQ^{-1}$ where $Q \in GL(2, \mathbb{C})$. A monomorphism $\psi : \mathbb{D}_N \to \operatorname{Aut} sl(2, \mathbb{C})$ is nothing but a faithful projective representation of $\mathbb{D}_N$ and it is sufficient to define it on the generators of the group. Let Q_s and Q_t correspond to σ_s and σ_t, respectively. Two projective representations Q_s, Q_t and $\hat{Q}_s, \hat{Q}_t$ are equivalent if there exist $W \in GL(2, \mathbb{C})$ and $c_s, c_t \in \mathbb{C}$ such that $WQ_sW^{-1} = c_s\hat{Q}_s$ and $WQ_tW^{-1} = c_t\hat{Q}_t$.

In the simplest case $\mathcal{G} = \mathbb{D}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there is only one class of faithful projective representations which is equivalent to the choice

$$Q_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

Thus the reduction group $\mathbb{D}_2$ can be generated by two transformations

$$g_s : \ a(\lambda) \to Q_s a(-\lambda) Q_s^{-1}, \quad g_t : \ a(\lambda) \to Q_t a(\lambda^{-1}) Q_t^{-1}, \quad a(\lambda) \in \mathcal{A}(\Gamma)$$

and the group average is

$$\langle a(\lambda) \rangle_{\mathbb{D}_2} = \frac{1}{4} (a(\lambda) + Q_s a(-\lambda) Q_s^{-1} + Q_t a(\lambda^{-1}) Q_t^{-1} + Q_t Q_s a(-\lambda^{-1}) Q_s^{-1} Q_t^{-1}).$$

In $\mathcal{A} = sl(2, \mathbb{C})$ we take the standard basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{18}$$

with commutation relations

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

Let $\gamma \in \hat{\mathbb{C}}$ be a generic point, i.e. $\gamma \notin \{0, \infty, \pm 1, \pm i\}$ and $|\mathcal{G}_\gamma| = 1$, then

$$x_\gamma(\lambda) = \langle \frac{x}{\lambda - \gamma} \rangle_{\mathbb{D}_2} = \begin{pmatrix} 0 & \frac{\lambda}{2(\lambda^2 - \gamma^2)} \\ \frac{\lambda}{2(1 - \lambda^2 \gamma^2)} & 0 \end{pmatrix} \tag{19}$$

$$y_\gamma(\lambda) = \langle \frac{y}{\lambda - \gamma} \rangle_{\mathbb{D}_2} = \begin{pmatrix} 0 & \frac{\lambda}{2(1 - \lambda^2 \gamma^2)} \\ \frac{\lambda}{2(\lambda^2 - \gamma^2)} & 0 \end{pmatrix} \tag{20}$$

$$h_\gamma(\lambda) = \langle \frac{h}{\lambda - \gamma} \rangle_{\mathbb{D}_2} = \frac{\gamma(1 - \lambda^4)}{2(\lambda^2 - \gamma^2)(1 - \lambda^2 \gamma^2)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

We shall denote $sl_{\mathbb{D}_2}(2, \mathbb{C}; \gamma)$ the infinite dimensional Lie algebra of all $\mathbb{D}_2$-automorphic traceless 2×2 matrices whose entries are rational functions in λ with poles at $\mathbb{D}_2(\gamma)$ and with no other singularities.

Proposition 3.1 *Let $\mu \in \mathbb{C} \setminus \{\pm\gamma, \pm\gamma^{-1}\}$. The set*

$$\begin{aligned} x_{\gamma\mu}^n &= 4x_\gamma(\lambda)(f_{\mathbb{D}_2}(\lambda, \gamma, \mu))^n \\ y_{\gamma\mu}^n &= 4y_\gamma(\lambda)(f_{\mathbb{D}_2}(\lambda, \gamma, \mu))^n \quad , \qquad n = 0, 1, 2, \ldots \\ h_{\gamma\mu}^n &= 4h_\gamma(\lambda)(f_{\mathbb{D}_2}(\lambda, \gamma, \mu))^n \end{aligned} \tag{22}$$

is a basis in $sl_{\mathbb{D}_2}(2, \mathbb{C}; \gamma)$. Here $f_{\mathbb{D}_2}(\lambda, \gamma, \mu)$ is a primitive automorphic function defined as

$$f_{\mathbb{D}_2}(\lambda, \gamma, \mu) = \alpha \frac{(\lambda^2 - \mu^2)(1 - \mu^2 \lambda^2)}{(\lambda^2 - \gamma^2)(1 - \gamma^2 \lambda^2)}, \quad \alpha = \frac{2\gamma(\gamma^4 - 1)}{(\mu^2 - \gamma^2)(1 - \mu^2 \gamma^2)}. \tag{23}$$

In (23) we have chosen the constant α to make $res_{\lambda=\gamma} f_{\mathbb{D}_2}(\lambda, \gamma, \mu) = 1$.

Proof We prove the proposition by induction. Let $a(\lambda) \in sl_{\mathbb{D}_2}(2, \mathbb{C}; \gamma)$. If $a(\lambda) = a_0$ does not have a singularity at $\lambda = \gamma$, then $a_0 = 0$. Indeed, in this case a_0 does not have singularities at all and therefore it is a constant matrix. It follows from $g_s(a_0) = a_0$ and $g_t(a_0) = a_0$ that a_0 commutes with Q_s and Q_t, therefore a_0 has to be proportional to the unit matrix. From $trace(a_0) = 0$ follows that $a_0 = 0$. Suppose $a(\lambda)$ has a pole of order $n > 0$ at $\lambda = \gamma$, then near the singularity it can be represented as $a(\lambda) = a_0(\lambda - \gamma)^{-n} + \hat{a}(\lambda)$ where a_0 is a constant matrix, $\hat{a}(\lambda)$ may have a pole at $\lambda = \gamma$ of order $m < n$. In the basis (18) $a_0 = c_1 x + c_2 y + c_3 h$, $c_i \in \mathbb{C}$. If

$$b(\lambda) = a(\lambda) - 4(c_1 x_\gamma(\lambda) + c_2 y_\gamma(\lambda) + c_3 h_\gamma(\lambda)) f_{\mathbb{D}_2}^{n-1}(\lambda, \gamma, \mu) \in sl_{\mathbb{D}_2}(2, \mathbb{C}, \gamma)$$

is singular at $\lambda = \gamma$ then the order of its pole is less or equal to $n - 1$ and this complete the induction step. $\square$

Proposition 3.2 *Elements $x_\gamma(\lambda), y_\gamma(\lambda), h_\gamma(\lambda)$ generates a $\mathbb{D}_2$-automorphic Lie algebra $sl_{\mathbb{D}_2}(2, \mathbb{C}; \gamma)$. The algebra $sl_{\mathbb{D}_2}(2, \mathbb{C}; \gamma)$ is quasigraded, its quasigraded structure*

$$sl_{\mathbb{D}_2}(2, \mathbb{C}; \gamma) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_\gamma^n(\mu), \qquad [\mathcal{L}_\gamma^n(\mu), \mathcal{L}_\gamma^m(\mu)] \subseteq \mathcal{L}_\gamma^{n+m+1}(\mu) \bigoplus \mathcal{L}_\gamma^{n+m}(\mu)$$

depends on a complex parameter μ and $\mathcal{L}_\gamma^n(\mu) = Span_\mathbb{C}(x_{\gamma\mu}^n, y_{\gamma\mu}^n, h_{\gamma\mu}^n)$.

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Proof Indeed, by direct calculation we find that

$$
\begin{aligned}
\left[x_{\gamma\mu}^{n}, y_{\gamma\mu}^{m}\right] &= h_{\gamma\mu}^{n+m+1} + a_{\gamma\mu}h_{\gamma\mu}^{n+m} \\
\left[h_{\gamma\mu}^{n}, x_{\gamma\mu}^{m}\right] &= 2x_{\gamma\mu}^{n+m+1} + b_{\gamma\mu}x_{\gamma\mu}^{n+m} - c_{\gamma\mu}y_{\gamma\mu}^{n+m}, \qquad n, m = 0, 1, 2, \dots \\
\left[h_{\gamma\mu}^{n}, y_{\gamma\mu}^{m}\right] &= -2y_{\gamma\mu}^{n+m+1} - b_{\gamma\mu}y_{\gamma\mu}^{n+m} + c_{\gamma\mu}x_{\gamma\mu}^{n+m}
\end{aligned} \tag{24}
$$

where

$$
a_{\gamma\mu} = \frac{2\mu^2(1-\gamma^4)}{\gamma(\mu^2-\gamma^2)(1-\mu^2\gamma^2)}, \quad b_{\gamma\mu} = \frac{4\gamma(1+\mu^4-4\mu^2\gamma^2+\gamma^4+\gamma^4\mu^4)}{(1-\gamma^4)(\mu^2-\gamma^2)(1-\mu^2\gamma^2)}, \quad c_{\gamma\mu} = \frac{8\gamma}{1-\gamma^4}.
$$

Thus, any element of the basis (22) can be generated by the set (19)–(21). It follows from (24) that $q = 0, p = 1$ (see (14)). $\square$

The quasigraded structure of $sl_{\mathbb{D}_2}(2,\mathbb{C};\gamma)$, i.e its decomposition in a direct sum of linear subspaces $\mathcal{L}_\gamma^n(\mu)$, depends on a complex parameter μ. This parameter determines the zeros of the primitive automorphic function $f_{\mathbb{D}_2}(\lambda,\gamma,\mu)$. Taking into account the fact that $f_{\mathbb{D}_2}(\lambda,\gamma,\nu) = f_{\mathbb{D}_2}(\lambda,\gamma,\mu) - f_{\mathbb{D}_2}(\nu,\gamma,\mu)$, we see that the corresponding bases $\{x_{\gamma\mu}^n, y_{\gamma\mu}^n, h_{\gamma\mu}^n\}_{n\in\mathbb{Z}_+}$ and $\{x_{\gamma\nu}^n, y_{\gamma\nu}^n, h_{\gamma\nu}^n\}_{n\in\mathbb{Z}_+}$ are related by a simple invertible triangular transformation

$$
x_{\gamma\nu}^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (f_{\mathbb{D}_2}(\nu,\gamma,\mu))^k \, x_{\gamma\mu}^{n-k} \tag{25}
$$

(same for $y_{\gamma\nu}^n$ and $h_{\gamma\nu}^n$), where $\binom{n}{k}$ are binomial coefficients. For positive n the sum (25) is finite, since all $\binom{n}{k}$ vanish as $k > n$.

The set $\{x_{\gamma\mu}^n, y_{\gamma\mu}^n, h_{\gamma\mu}^n\}$ is naturally defined for negative integers $n \in \mathbb{Z}_-$.

Proposition 3.3 *Elements* $x_{\gamma\mu}^{-1}, y_{\gamma\mu}^{-1}, h_{\gamma\mu}^{-1}$ *generate a* $\mathbb{D}_2$*-automorphic Lie algebra* $sl_{\mathbb{D}_2}(2,\mathbb{C};\mu)$. *The set* $\{x_{\gamma\mu}^n, y_{\gamma\mu}^n, h_{\gamma\mu}^n \,|\, n \in \mathbb{Z}_-\}$ *is a basis in* $sl_{\mathbb{D}_2}(2,\mathbb{C},\mu)$.

Proof For negative n, automorphic elements $x_{\gamma\mu}^n, y_{\gamma\mu}^n, h_{\gamma\mu}^n$ have poles at points $\mathcal{G}(\mu)$ and do not have other singularities, therefore $x_{\gamma\mu}^n, y_{\gamma\mu}^n, h_{\gamma\mu}^n \in sl_{\mathbb{D}_2}(2,\mathbb{C};\mu)$. The proof that $\{x_{\gamma\mu}^n, y_{\gamma\mu}^n, h_{\gamma\mu}^n \,|\, n \in \mathbb{Z}_-\}$ form a basis in $sl_{\mathbb{D}_2}(2,\mathbb{C};\mu)$ is similar to Proposition 3.1. $\square$

Thus, with any two orbits $\mathbb{D}_2(\gamma)$ and $\mathbb{D}_2(\mu)$ we associate two uniquely defined subalgebras $sl_{\mathbb{D}_2}(2,\mathbb{C},\gamma)$ and $sl_{\mathbb{D}_2}(2,\mathbb{C},\mu)$ of the infinite dimensional Lie algebra

$$
sl_{\mathbb{D}_2}(2,\mathbb{C};\gamma,\mu) = sl_{\mathbb{D}_2}(2,\mathbb{C},\gamma) \bigoplus sl_{\mathbb{D}_2}(2,\mathbb{C},\mu).
$$

The set (22) with $n \in \mathbb{Z}$ is a basis in $sl_{\mathbb{D}_2}(2,\mathbb{C};\gamma,\mu)$ with commutation relations (24). $sl_{\mathbb{D}_2}(2,\mathbb{C};\gamma,\mu)$ has a uniquely defined quasigraded structure corresponding to a primitive automorphic function $f_{\mathbb{D}_2}(\lambda,\gamma,\mu)$. Quasigraded automorphic algebras corresponding to different orbits are not isomorphic, i.e. elements of one algebra cannot be represented by finite linear combination of the basis elements of the other algebra with complex constant coefficients.

In the above construction, the point μ could be a generic point or belong to one of the degenerated orbits. Having generators and structure constants for algebra $sl_{\mathbb{D}_2}(2,\mathbb{C},\gamma)$ we can easily find generators and corresponding structure constants for $sl_{\mathbb{D}_2}(2,\mathbb{C},\mu)$. Taking, for example, $\mu = 0$ we find generators

$$
\hat{x}_0 = 4x_\gamma(f_{\mathbb{D}_2}(\lambda,\gamma,0))^{-1}, \quad \hat{y}_0 = 4y_\gamma(f_{\mathbb{D}_2}(\lambda,\gamma,0))^{-1}, \quad \hat{h}_0 = 4h_\gamma(f_{\mathbb{D}_2}(\lambda,\gamma,0))^{-1} \tag{26}
$$

for $sl_{\mathbb{D}_2}(2,\mathbb{C},0)$. The set $\{\hat{x}_0^n = \hat{x}_0(f_{\mathbb{D}_2}(\lambda,\gamma,0))^{-n}, \hat{y}_0^n = \hat{y}_0(f_{\mathbb{D}_2}(\lambda,\gamma,0))^{-n}, \hat{h}_0^n = \hat{h}_0(f_{\mathbb{D}_2}(\lambda,\gamma,0))^{-n} | n \in \mathbb{Z}_+\}$ can be taken as a basis (compare with Proposition 3.3). The structure constants in this basis follows immediately from (24).

The generators of $sl_{\mathbb{D}_2}(2,\mathbb{C},0)$ can also be found directly, by taking the group average

$$x_0(\lambda) = \langle \frac{x}{\lambda} \rangle_{\mathbb{D}_2} = \frac{1}{2} \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix},$$

$$y_0(\lambda) = \langle \frac{y}{\lambda} \rangle_{\mathbb{D}_2} = \frac{1}{2} \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \tag{27}$$

$$h_0(\lambda) = \langle \frac{h}{\lambda^2} \rangle_{\mathbb{D}_2} = \frac{(1-\lambda^4)}{2\lambda^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Generators (26) can be expressed in terms of (27)

$$\hat{x}_0 = \frac{2\gamma}{1-\gamma^4}(x_0 - \gamma^2 y_0), \quad \hat{y}_0 = \frac{2\gamma}{1-\gamma^4}(y_0 - \gamma^2 x_0), \quad \hat{h}_0 = \frac{8\gamma^2}{1-\gamma^4}h_0.$$

In the basis $\{x_0^n = x_0 J^n, y_0^n = y_0 J^n, h_0^n = \frac{1}{2}h_0 J^n\}_{n \in \mathbb{Z}_+}$ where $J = f_{\mathbb{D}_2}(\lambda, 0, 1) = \frac{1}{2}(\lambda - \lambda^{-1})^2$, the commutation relations of $sl_{\mathbb{D}_2}(2,\mathbb{C},0)$ take a very simple form:

$$[x_0^n, y_0^m] = h_0^{n+m}, \quad [h_0^n, x_0^m] = x_0^{n+m+1} + x_0^{n+m} - y_0^{n+m}, \quad [h_0^n, y_0^m] = -y_0^{n+m+1} - y_0^{n+m} + x_0^{n+m}.$$

In the case $\mathcal{G} \cong \mathbb{D}_3$ the projective representation is generated by Q_t (17) and $Q_s = \mathrm{diag}\,(e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}})$. Using the group average one can find $sl_{\mathbb{D}_3}(2,\mathbb{C},\gamma)$ algebra generators and then the basis in which the algebra has a quasigraded structure. It turns out that algebra $sl_{\mathbb{D}_3}(2,\mathbb{C},\gamma)$ is isomorphic to $sl_{\mathbb{D}_2}(2,\mathbb{C},\mu)$ if $\gamma^3 = \mu^2$. In particularly $sl_{\mathbb{D}_3}(2,\mathbb{C},0) \cong sl_{\mathbb{D}_2}(2,\mathbb{C},0)$. It is a general observation – for any $N, M \geq 2$ and $\gamma \in \mathbb{C}$

$$sl_{\mathbb{D}_N}(2,\mathbb{C},\gamma^M) \cong sl_{\mathbb{D}_M}(2,\mathbb{C},\gamma^N).$$

For $N > 2$ there is a choice of inequivalent irreducible representations of $\mathbb{D}_N$. Automorphic Lie algebras corresponding to different representations proved to be isomorphic. This explains why integrable equations corresponding to $\mathbb{D}_N$ reductions with different N and non equivalent representations coincide [14].

3.2 Automorphic Lie algebras with $\mathcal{G} = \mathbb{D}_N$, $\mathcal{A} = sl(3,\mathbb{C})$.

Let the action of $\mathbb{D}_N$ on the complex plane λ be the same as in the previous section, i.e. generated by two fractional-linear transformations $\sigma_s(\lambda) = \Omega\lambda$ and $\sigma_t(\lambda) = \lambda^{-1}$ with $\Omega = \exp(2i\pi/N)$.

It follows from Theorem 2.3 that automorphisms $\mathrm{Aut}\,(sl(3,\mathbb{C}))$ can be represented either in the form $a \to QaQ^{-1}$ or $a \to -Ha^{tr}H^{-1}$ where $Q, H \in GL(3,\mathbb{C})$. The first kind of automorphisms (with Q) form a normal subgroup of inner automorphisms $\mathrm{Aut}\,_0(sl(3,\mathbb{C}))$, while automorphisms with H correspond to outer automorphisms and $\mathrm{Aut}\,(sl(3,\mathbb{C}))/\mathrm{Aut}\,_0(sl(3,\mathbb{C})) \cong \mathbb{Z}_2$. There are two distinct ways to define a monomorphism $\psi : \mathbb{D}_N \to \mathrm{Aut}\,(sl(3,\mathbb{C}))$:

Case A ψ maps $\mathbb{D}_N$ into the subgroup of inner automorphisms (similar to the previous section). In this case ψ is nothing but a faithful projective representation of $\mathbb{D}_N$.

Case B The other option is to use a normal subgroup decomposition $(id \to \mathbb{Z}_N \to \mathbb{D}_N \to \mathbb{Z}_2 \to id)$. In this case ψ maps the normal subgroup $\mathbb{Z}_N$ in $\mathrm{Aut}\,_0(sl(3,\mathbb{C}))$, and its co-set into the co-set corresponding to outer automorphisms, so that the following commutative diagram is exact:

$$\begin{array}{ccccccccc}
& & id & & id & & id & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
id & \longrightarrow & \mathbb{Z}_N & \longrightarrow & \mathbb{D}_N & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & id \\
& & \downarrow \psi & & \downarrow \psi & & \downarrow & & \\
id & \longrightarrow & \mathrm{Aut}\,_0(sl(3,\mathbb{C})) & \longrightarrow & \mathrm{Aut}\,(sl(3,\mathbb{C})) & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & id
\end{array}$$

We shall study these two cases separately.

3.2.1 Case A. Inner automorphisms representation.

We shall see that in the case $\mathcal{A} = sl(3,\mathbb{C})$ the reduction groups $\mathbb{D}_2^A$ and $\mathbb{D}_N^A$, $N > 2$ yield non-isomorphic automorphic Lie algebras (the upper index stands for the case A). Let $\mathbf{e}_{ij}$ denotes a matrix with 1 at the position (i,j) and zeros elsewhere. Matrices $\mathbf{e}_{ij}$, $i \neq j$ and $h_1 = \mathbf{e}_{11} - \mathbf{e}_{22}$, $h_2 = \mathbf{e}_{22} - \mathbf{e}_{33}$ form a basis in $sl(3,\mathbb{C})$.

Case A, $G = \mathbb{D}_2^A$: The action of the reduction group $G = \mathbb{D}_2^A$ can be generated by transformations

$$g_s : a(\lambda) \to Q_s a(-\lambda) Q_s^{-1}, \quad g_t : a(\lambda) \to Q_t a(1/\lambda) Q_t^{-1}, \tag{28}$$

where $Q_s = \mathrm{diag}\,(-1,1,-1)$ and $Q_t = \mathrm{diag}\,(1,-1,-1)$. It is easy to check that $g_s^2 = g_t^2 = (g_s g_t)^2 = id$. If one ignores the λ transformations, then (28) form a $\mathbb{D}_2$ subgroup of inner automorphisms of algebra $sl(3,\mathbb{C})$.

In order to fix a primitive automorphic Lie algebra, we need to fix two orbits of the reduction group on the complex plane λ. As in the previous section, the choice of the orbits is not very essential, since knowing the structure constants of the algebra for one choice of the orbits, we can easily reconstruct the structure constants for any other choice. We shall consider the orbits $\{0,\infty\}$ and $\{1,-1\}$ and take the corresponding primitive automorphic function in the form

$$J = f_{\mathbb{D}_2}(\lambda, 0, 1) = \lambda^2 + \lambda^{-2} - 2\,.$$

Automorphic Lie algebra $sl_{\mathbb{D}_2^A}(3,\mathbb{C};0,1)$ is quasigraded ($[\mathcal{A}_n, \mathcal{A}_m] \subset \mathcal{A}_{n+m} \bigoplus \mathcal{A}_{n+m+1} \bigoplus \mathcal{A}_{n+m+2}$),

$$sl_{\mathbb{D}_2^A}(3,\mathbb{C};0,1) = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n\,,$$

where $\mathcal{A}_n = J^n \mathcal{A}_0$. It is sufficient to give a description of the linear space $\mathcal{A}_0$ and commutation relations $[\mathcal{A}_0, \mathcal{A}_0]$. A basis in the eight dimensional space $\mathcal{A}_0$ can be chosen as:

$$
\begin{aligned}
x_1^0 &= \langle 2\mathbf{e}_{12}\lambda^{-1}\rangle_{\mathbb{D}_2^A} = (\lambda^{-1} - \lambda)\mathbf{e}_{12}\,, & y_1^0 &= \langle 2\mathbf{e}_{21}\lambda^{-1}\rangle_{\mathbb{D}_2^A} = (\lambda^{-1} - \lambda)\mathbf{e}_{21} \\
x_2^0 &= \langle 2\mathbf{e}_{23}\lambda^{-1}\rangle_{\mathbb{D}_2^A} = (\lambda^{-1} + \lambda)\mathbf{e}_{23}\,, & y_2^0 &= \langle 2\mathbf{e}_{32}\lambda^{-1}\rangle_{\mathbb{D}_2^A} = (\lambda^{-1} + \lambda)\mathbf{e}_{32} \\
x_3^0 &= [x_1^0, x_2^0] = (\lambda^{-2} - \lambda^2)\mathbf{e}_{13}\,, & y_3^0 &= [y_2^0, y_1^0] = (\lambda^{-2} - \lambda^2)\mathbf{e}_{31}
\end{aligned}
\tag{29}
$$

$$h_1^0 = \mathbf{e}_{11} - \mathbf{e}_{22}\,, \qquad h_2^0 = \mathbf{e}_{22} - \mathbf{e}_{33}\,. \tag{30}$$

Proposition 3.4 *The set*

$$x_i^n = J^n x_i^0\,, \quad y_i^n = J^n y_i^0\,, \quad h_j^n = J^n h_j^0\,, \quad i \in \{1,2,3\}, \quad j \in \{1,2\}, \quad n \in \mathbb{Z} \tag{31}$$

is a basis of the algebra $sl_{\mathbb{D}_2^A}(3,\mathbb{C};0,1)$.

The proof is similar to Propositions 3.1, 3.3. It is easy to compute all commutators between the basis elements of $sl_{\mathbb{D}_2^A}(3,\mathbb{C};0,1)$. For example

$$[h_1^n, x_1^m] = 2x_1^{n+m}\,, \quad [x_1^n, y_1^m] = h_1^{n+m+1} - 2h_1^{n+m}\,, \quad [x_3^n, y_3^m] = h_1^{n+m+2} + h_2^{n+m+2} - 4h_1^{n+m} - 4h_2^{n+m}\,.$$

Algebra $sl_{\mathbb{D}_2^A}(3,\mathbb{C};0,1)$ has a quasigraded subalgebra $\hat{sl}_{\mathbb{D}_2^A}(3,\mathbb{C};0,1)$, which is a direct sum of two infinite dimensional subalgebras

$$\hat{sl}_{\mathbb{D}_2^A}(3,\mathbb{C};0,1) = \hat{sl}_{\mathbb{D}_2^A}(3,\mathbb{C};0) \bigoplus \hat{sl}_{\mathbb{D}_2^A}(3,\mathbb{C};1)$$

As a basis in $\hat{sl}_{\mathbb{D}_2^A}(3,\mathbb{C};0,1)$ we can take a set x_i^n, y_i^n defined in (29), (31) and

$$\hat{h}_1^n = J^n[x_1^0, y_1^0] = (\lambda^2 + \lambda^{-2} - 2)J^n(\mathbf{e}_{11} - \mathbf{e}_{22}), \quad \hat{h}_2^n = J^n[x_2^0, y_2^0] = (\lambda^2 + \lambda^{-2} + 2)J^n(\mathbf{e}_{22} - \mathbf{e}_{33}), \qquad n \in \mathbb{Z}.$$

In this basis the non-vanishing commutation relations are

$$
\begin{array}{lll}
[\hat{h}_1^n, x_1^m] = 2x_1^{n+m+1}, & [\hat{h}_1^n, y_1^m] = -2y_1^{n+m+1}, & [\hat{h}_1^n, x_2^m] = -x_2^{n+m+1}, \\
[\hat{h}_1^n, y_2^m] = y_2^{n+m+1}, & [\hat{h}_1^n, x_3^m] = x_3^{n+m+1}, & [\hat{h}_1^n, y_3^m] = -y_3^{n+m+1}, \\
[\hat{h}_2^n, x_1^m] = -x_1^{n+m+1} - 4x_1^{n+m}, & [\hat{h}_2^n, y_1^m] = y_1^{n+m+1} + 4y_1^{n+m}, & [\hat{h}_2^n, x_2^m] = 2x_2^{n+m+1} + 8x_2^{n+m}, \\
[\hat{h}_2^n, y_2^m] = -2y_2^{n+m+1} - 8y_2^{n+m}, & [\hat{h}_2^n, x_3^m] = x_3^{n+m+1} + 4x_3^{n+m}, & [\hat{h}_2^n, y_3^m] = -y_3^{n+m+1} - 4y_3^{n+m}, \\
[x_1^n, x_2^m] = x_3^{n+m}, & [y_1^n, y_2^m] = -y_3^{n+m}, & [x_1^n, y_1^m] = \hat{h}_1^{n+m}, \\
[x_1^n, y_3^m] = -y_2^{n+m+1}, & [x_2^n, y_2^m] = \hat{h}_2^{n+m}, & [x_2^n, y_3^m] = y_1^{n+m+1} + 4y_1^{n+m}, \\
[x_3^n, y_1^m] = -x_2^{n+m+1}, & [x_3^n, y_2^m] = x_1^{n+m+1} + 4x_1^{n+m}, & [x_3^n, y_3^m] = \hat{h}_1^{n+m+1} + \hat{h}_2^{n+m+1} + 4\hat{h}_1^{n+m}.
\end{array}
$$

Elements $x_i^n, y_i^n, \hat{h}_j^n$ with $n \geq 0$ form a basis in $\hat{sl}_{\mathbb{D}_2^A}(3,\mathbb{C};0)$, elements with $n < 0$ form a basis in $\hat{sl}_{\mathbb{D}_2^A}(3,\mathbb{C};1)$.

Case A, $G = \mathbb{D}_3^A$: The action of the reduction group $G = \mathbb{D}_3^A$ can be generated by transformations

$$g_s : a(\lambda) \to Q_s a(\omega^{-1}\lambda)Q_s^{-1}, \quad g_t : a(\lambda) \to Q_t a(1/\lambda)Q_t^{-1}, \tag{32}$$

where

$$Q_s = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix}, \quad \omega = \exp\left(\frac{2\pi i}{3}\right)$$

It is easy to check that $g_s^3 = g_t^2 = (g_s g_t)^2 = id$. The signs in Q_t correspond to two inequivalent representations of $\mathbb{D}_3^A$.

Let us choose the following automorphic function

$$f = \lambda^3 + \lambda^{-3}, \tag{33}$$

corresponding to the orbits $\mathbb{D}_3(0) = \{0, \infty\}$ and $\mathbb{D}_3(\varpi)$ where $\varpi = \exp(\pi i/6)$. The automorphic Lie algebra $sl_{\mathbb{D}_3^A}(3,\mathbb{C};0,\varpi) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$ is quasigraded. A basis in the eight dimensional space $\mathcal{A}_0$ can be chosen as:

$$
\begin{array}{ll}
x_1^0 = \langle 2\mathbf{e}_{12}\lambda^{-1}\rangle_{\mathbb{D}_3^A} = \lambda^{-1}\mathbf{e}_{12} + \lambda\mathbf{e}_{21}, & y_1^0 = \langle 2\mathbf{e}_{21}\lambda^{-2}\rangle_{\mathbb{D}_3^A} = \lambda^{-2}\mathbf{e}_{21} + \lambda^2\mathbf{e}_{12} \\
x_2^0 = \langle 4\mathbf{e}_{23}\lambda^{-1}\rangle_{\mathbb{D}_3^A} = 2\lambda^{-1}\mathbf{e}_{23} \mp 2\lambda\mathbf{e}_{13}, & y_2^0 = \langle 4\mathbf{e}_{32}\lambda^{-2}\rangle_{\mathbb{D}_3^A} = 2\lambda^{-2}\mathbf{e}_{32} \mp \lambda^2\mathbf{e}_{31} \\
x_3^0 = [x_1^0, x_2^0] = 2\lambda^{-2}\mathbf{e}_{13} \pm 2\lambda^2\mathbf{e}_{23}, & y_3^0 = \langle 4\mathbf{e}_{31}\lambda^{-1}\rangle_{\mathbb{D}_3^A} = 2\lambda^{-1}\mathbf{e}_{31} \mp 2\lambda\mathbf{e}_{32}
\end{array} \tag{34}
$$

$$h_1^0 = \langle 2(\mathbf{e}_{11} - \mathbf{e}_{22})\lambda^{-3}\rangle_{\mathbb{D}_3^A} = (\lambda^{-3} - \lambda^3)(\mathbf{e}_{11} - \mathbf{e}_{22}), \quad h_2^0 = \frac{2}{3}(\mathbf{e}_{11} + \mathbf{e}_{22} - 2\mathbf{e}_{33}). \tag{35}$$

In the basis

$$x_i^n = f^n x_i^0, \quad y_i^n = f^n y_i^0, \quad h_j^n = f^n h_j^0, \quad i \in \{1,2,3\}, \quad j \in \{1,2\}, \quad n \in \mathbb{Z} \tag{36}$$

of the automorphic Lie algebra $sl_{\mathbb{D}_3^A}(3,\mathbb{C};0,\varpi)$ the non-vanishing commutation relations are ($n, m \in \mathbb{Z}$)

$$
\begin{array}{lll}
[h_1^n, x_1^m] = 2x_1^{n+m+1} - 4y_1^{n+m}, & [h_1^n, y_1^m] = -2y_1^{n+m+1} + 4x_1^{n+m}, & [h_1^n, x_2^m] = -x_2^{n+m+1} \mp 2x_3^{n+m}, \\
[h_1^n, y_2^m] = y_2^{n+m+1} \pm 2y_3^{n+m}, & [h_1^n, x_3^m] = x_3^{n+m+1} \pm 2x_2^{n+m}, & [h_1^n, y_3^m] = -y_3^{n+m+1} \mp 2y_2^{n+m}, \\
[h_2^n, x_2^m] = 2x_2^{n+m}, & [h_2^n, y_2^m] = -2y_2^{n+m}, & [h_2^n, x_3^m] = 2x_3^{n+m}, \\
[h_2^n, y_3^m] = -2y_3^{n+m}, & [x_1^n, x_2^m] = x_3^{n+m}, & [x_1^n, x_3^m] = x_2^{n+m}, \\
[y_1^n, y_2^m] = -y_3^{n+m+1} \mp y_2^{n+m}, & [y_1^n, y_3^m] = \pm y_3^{n+m}, & [x_1^n, y_1^m] = h_1^{n+m}, \\
[x_1^n, y_2^m] = -y_3^{n+m}, & [x_1^n, y_3^m] = -y_2^{n+m}, & [x_2^n, y_1^m] = \pm x_2^{n+m}, \\
[x_2^n, y_2^m] = 3h_2^{n+m+1} - 2h_1^{n+m} \mp 4x_1^{n+m}, & [x_2^n, y_3^m] = \mp 6h_2^{n+m} + 4y_1^{n+m}, & [x_3^n, y_1^m] = -x_2^{n+m+1} \mp x_3^{n+m}, \\
[x_3^n, y_2^m] = 4x_1^{n+m+1} - 4y_1^{n+m} \mp 6h_2^{n+m}, & [x_3^n, y_3^m] = 3h_2^{n+m+1} + 2h_1^{n+m} \mp 4x_1^{n+m}.
\end{array}
$$

A subset of (36) with $n \geq 0$ form a basis of the subalgebra $sl_{\mathbb{D}_3^A}(3,\mathbb{C};0)$, while elements with $n < 0$ are a basis of the subalgebra $sl_{\mathbb{D}_3^A}(3,\mathbb{C};\varpi)$, and it follows from the above commutation relations that algebra $sl_{\mathbb{D}_3^A}(3,\mathbb{C};0,\varpi)$ is a direct sum of these subalgebras.

3.2.2 Case B. Inner and outer automorphisms representation.

Reduction groups $\mathbb{D}_2^B$ and $\mathbb{D}_N^B$ with $N > 2$ yield different automorphic Lie algebras and we consider these sub-cases separately. In the both sub-cases we shall use a primitive automorphic function $f = \lambda^N + \lambda^{-N}$.

Case B, $G = \mathbb{D}_2^B$: The action of the reduction group $G = \mathbb{D}_2^B$ can be generated by two transformations

$$g_s : a(\lambda) \to Q_s a(-\lambda) Q_s^{-1}, \quad g_t : a(\lambda) \to -a^{tr}(1/\lambda), \tag{37}$$

where $Q_s = \mathrm{diag}\,(-1, -1, 1)$. Indeed, these transformations generates the group $\mathbb{D}_2$, is easy to check that $g_s^2 = g_t^2 = (g_s g_t)^2 = id$. If one ignores the λ transformations (i.e. takes π_2 natural projection), then the first transformations in (37) is an inner automorphism of algebra $sl(3, \mathbb{C})$, while the second one is an outer automorphism.

The corresponding automorphic Lie algebra $sl_{\mathbb{D}_2^B}(3, \mathbb{C}; 0, \exp(i\pi/4))$ has a basis of the form (36) where

$$
\begin{aligned}
x_1^0 &= \langle 2\mathbf{e}_{12}\rangle_{\mathbb{D}_2^B} = \mathbf{e}_{12} - \mathbf{e}_{21}, & y_1^0 &= \langle 2\mathbf{e}_{21}l^{-2}\rangle_{\mathbb{D}_2^B} = \lambda^{-2}\mathbf{e}_{21} - \lambda^2 \mathbf{e}_{12}, \\
x_2^0 &= \langle 2\mathbf{e}_{23}\lambda^{-1}\rangle_{\mathbb{D}_2^B} = \lambda^{-1}\mathbf{e}_{23} - \lambda \mathbf{e}_{32}, & y_2^0 &= \langle 2\mathbf{e}_{32}\lambda^{-1}\rangle_{\mathbb{D}_2^B} = \lambda^{-1}\mathbf{e}_{32} - \lambda \mathbf{e}_{23}, \\
x_3^0 &= \langle 2\mathbf{e}_{13}\lambda^{-1}\rangle_{\mathbb{D}_2^B} = \lambda^{-1}\mathbf{e}_{13} - \lambda \mathbf{e}_{31}, & y_3^0 &= \langle 2\mathbf{e}_{31}\lambda^{-1}\rangle_{\mathbb{D}_2^B} = \lambda^{-1}\mathbf{e}_{31} - \lambda \mathbf{e}_{13},
\end{aligned} \tag{38}
$$

$$h_1^0 = \langle 2(\mathbf{e}_{11} - \mathbf{e}_{22})\lambda^{-2}\rangle_{\mathbb{D}_2^B} = (\lambda^{-2} - \lambda^2)(\mathbf{e}_{11} - \mathbf{e}_{22}), \quad h_2^0 = \langle 2(\mathbf{e}_{22} - \mathbf{e}_{33})\lambda^{-2}\rangle_{\mathbb{D}_2^B} = (\lambda^{-2} - \lambda^2)(\mathbf{e}_{22} - \mathbf{e}_{33}).$$

The nonvanishing commutation relations of the automorphic Lie algebra $sl_{\mathbb{D}_2^B}(3, \mathbb{C}; 0, \exp \pi i/4)$ are

$$
\begin{array}{lll}
[h_1^n, x_1^m] = 2x_1^{n+m+1} + 4y_1^{n+m}, & [h_1^n, y_1^m] = -2y_1^{n+m+1} - 4x_1^{n+m}, & [h_1^n, x_2^m] = -x_2^{n+m+1} - 2y_2^{n+m}, \\
[h_1^n, y_2^m] = y_2^{n+m+1} + 2x_2^{n+m}, & [h_1^n, x_3^m] = x_3^{n+m+1} + 2y_3^{n+m}, & [h_1^n, y_3^m] = -y_3^{n+m+1} - 2x_3^{n+m}, \\
[h_2^n, x_1^m] = -x_1^{n+m+1} - 2y_1^{n+m}, & [h_2^n, y_1^m] = y_1^{n+m+1} + 2x_1^{n+m}, & [h_2^n, x_2^m] = 2x_2^{n+m+1} + 4y_2^{n+m}, \\
[h_2^n, y_2^m] = -2y_2^{n+m+1} - 4x_2^{n+m}, & [h_2^n, x_3^m] = x_3^{n+m+1} + 2y_3^{n+m}, & [h_2^n, y_3^m] = -y_3^{n+m+1} - 2x_3^{n+m}, \\
[x_1^n, y_1^m] = h_1^{n+m}, & [x_1^n, y_2^m] = y_3^{n+m}, & [x_1^n, y_3^m] = -y_2^{n+m}, \\
[x_1^n, x_2^m] = x_3^{n+m}, & [x_1^n, x_3^m] = -x_2^{n+m}, & [x_2^n, y_1^m] = -y_3^{n+m}, \\
[x_2^n, y_2^m] = h_2^{n+m}, & [x_2^n, y_3^m] = y_1^{n+m}, & [x_2^n, x_3^m] = x_1^{n+m}, \\
[x_3^n, y_1^m] = -x_2^{n+m+1} - y_2^{n+m}, & [x_3^n, y_2^m] = x_1^{n+m+1} + y_1^{n+m}, & [x_3^n, y_3^m] = h_2^{n+m} + h_1^{n+m}, \\
[y_1^n, y_2^m] = -y_3^{n+m+1} - x_3^{n+m}, & [y_1^n, y_3^m] = -x_2^{n+m}, & [y_2^n, y_3^m] = x_1^{n+m}.
\end{array}
$$

Case B, $G = \mathbb{D}_3^B$: The action of the reduction group $G = \mathbb{D}_3^B$ can be generated by transformations

$$g_s : a(\lambda) \to Q_s a(\omega^{-1}\lambda) Q_s^{-1}, \quad g_t : a(\lambda) \to -a^{tr}(1/\lambda), \tag{39}$$

where $Q_s = \mathrm{diag}\,(\omega, \omega^2, 1)$.

A basis of the algebra $sl_{\mathbb{D}_3^B}(3, \mathbb{C}; 0, \exp \pi i/6)$ has of the form (36) where

$$
\begin{aligned}
x_1^0 &= \langle 2\mathbf{e}_{12}\lambda^{-2}\rangle_{\mathbb{D}_3^B} = \lambda^{-2}\mathbf{e}_{12} - \lambda^2 \mathbf{e}_{21}, & y_1^0 &= \langle 2\mathbf{e}_{21}\lambda^{-1}\rangle_{\mathbb{D}_3^B} = \lambda^{-1}\mathbf{e}_{21} - \lambda \mathbf{e}_{12}, \\
x_2^0 &= \langle 2\mathbf{e}_{23}\lambda^{-2}\rangle_{\mathbb{D}_3^B} = \lambda^{-2}\mathbf{e}_{23} - \lambda^2 \mathbf{e}_{32}, & y_2^0 &= \langle 2\mathbf{e}_{32}\lambda^{-1}\rangle_{\mathbb{D}_3^B} = \lambda^{-1}\mathbf{e}_{32} - \lambda \mathbf{e}_{23}, \\
x_3^0 &= \langle 2\mathbf{e}_{13}\lambda^{-1}\rangle_{\mathbb{D}_3^B} = \lambda^{-1}\mathbf{e}_{13} - \lambda \mathbf{e}_{31}, & y_3^0 &= \langle 2\mathbf{e}_{31}\lambda^{-2}\rangle_{\mathbb{D}_3^B} = \lambda^{-2}\mathbf{e}_{31} - \lambda^2 \mathbf{e}_{13},
\end{aligned} \tag{40}
$$

$$h_1^0 = \langle 2(\mathbf{e}_{11} - \mathbf{e}_{22})\lambda^{-3}\rangle_{\mathbb{D}_3^B} = (\lambda^{-3} - \lambda^3)(\mathbf{e}_{11} - \mathbf{e}_{22}), \quad h_2^0 = \langle 2(\mathbf{e}_{22} - \mathbf{e}_{33})\lambda^{-3}\rangle_{\mathbb{D}_3^B} = (\lambda^{-3} - \lambda^3)(\mathbf{e}_{22} - \mathbf{e}_{33}).$$

The non-vanishing commutation relations of the automorphic Lie algebra $sl_{\mathbb{D}_3^B}(3, \mathbb{C}; 0, \exp \pi i/6)$ are

$$
\begin{array}{lll}
[h_1^n, x_1^m] = 2x_1^{n+m+1} + 4y_1^{n+m}, & [h_1^n, y_1^m] = -2y_1^{n+m+1} - 4x_1^{n+m}, & [h_1^n, x_2^m] = -x_2^{n+m+1} - 2y_2^{n+m}, \\
[h_1^n, y_2^m] = y_2^{n+m+1} + 2x_2^{n+m}, & [h_1^n, x_3^m] = x_3^{n+m+1} + 2y_3^{n+m}, & [h_1^n, y_3^m] = -y_3^{n+m+1} - 2x_3^{n+m}, \\
[h_2^n, x_1^m] = -x_1^{n+m+1} - 2y_1^{n+m}, & [h_2^n, y_1^m] = y_1^{n+m+1} + 2x_1^{n+m}, & [h_2^n, x_2^m] = 2x_2^{n+m+1} + 4y_2^{n+m}, \\
[h_2^n, y_2^m] = -2y_2^{n+m+1} - 4x_2^{n+m}, & [h_2^n, x_3^m] = x_3^{n+m+1} + 2y_3^{n+m}, & [h_2^n, y_3^m] = -y_3^{n+m+1} - 2x_3^{n+m}, \\
[x_1^n, y_1^m] = h_1^{n+m}, & [x_1^n, y_2^m] = -x_3^{n+m}, & [x_1^n, y_3^m] = -y_2^{n+m+1} - x_2^{n+m}, \\
[x_1^n, x_2^m] = x_3^{n+m+1} + y_3^{n+m}, & [x_1^n, x_3^m] = y_2^{n+m}, & [x_2^n, y_1^m] = x_3^{n+m}, \\
[x_2^n, y_2^m] = h_2^{n+m}, & [x_2^n, y_3^m] = y_1^{n+m+1} + x_1^{n+m}, & [x_2^n, x_3^m] = -y_1^{n+m}, \\
[x_3^n, y_1^m] = -x_2^{n+m}, & [x_3^n, y_2^m] = x_1^{n+m}, & [x_3^n, y_3^m] = h_2^{n+m} + h_1^{n+m}, \\
[y_1^n, y_2^m] = -y_3^{n+m}, & [y_1^n, y_3^m] = +y_2^{n+m}, & [y_2^n, y_3^m] = -y_1^{n+m}.
\end{array}
$$

It follows from the commutation relations that the automorphic Lie algebra $sl_{\mathbb{D}_2^B}(3,\mathbb{C};0,\exp\pi i/4)$ (similarly $sl_{\mathbb{D}_3^B}(3,\mathbb{C};0,\exp\pi i/6)$) is a direct sum of two subalgebras $sl_{\mathbb{D}_2^B}(3,\mathbb{C};0)$ and $\tilde{sl}_{\mathbb{D}_2^B}(3,\mathbb{C};\exp\pi i/4)$ (correspondingly $sl_{\mathbb{D}_3^B}(3,\mathbb{C};0)$ and $sl_{\mathbb{D}_3^B}(3,\mathbb{C};\exp\pi i/6)$). Basis elements with non-negative upper index form a basis of $sl_{\mathbb{D}_2^B}(3,\mathbb{C};0)$ $(sl_{\mathbb{D}_3^B}(3,\mathbb{C};0))$, while elements with negative index are a basis of $sl_{\mathbb{D}_2^B}(3,\mathbb{C};\exp\pi i/4)$ $(sl_{\mathbb{D}_3^B}(3,\mathbb{C};\exp\pi i/6))$. This algebra does not have constant (λ independent) elements.

3.3 Automorphic Lie algebras corresponding to *twisted* (λ-dependent) automorphisms.

In the previous sections we assumed that elements of the group $\operatorname{Aut}\mathcal{A}$ do not depend on the complex parameter λ. The group $\operatorname{Aut}\mathcal{A}$ is a continuous Lie group and we can admit that some of its elements depend on λ. In this case, the transformations of a reduction group G can be represented by pairs $(\sigma,\psi(\lambda))$, where the first element of the pair is a fractional-linear transformation of the complex plane λ while the second entry is a "λ–dependent" automorphism of the Lie algebra $\mathcal{A}$. To treat this case one needs to generalise the direct product of groups to the *semi-direct product* of groups [18], [19]:

Definition 3.5 *Let G_1, G_2 be two groups and ϕ be a homomorphism of G_1 into the group of automorphisms of G_2, denoted by $\operatorname{Aut}G_2$. Then $G_1 \times G_2$ with the product defined by*

$$(x,y)\cdot(x_1,y_1) = (x\cdot x_1, y\cdot\phi(x)y_1)$$

is a group called the semi-direct product *and denoted by $G_1 \times_\phi G_2$.*

When the homomorphism $\phi: G_1 \to \operatorname{Aut}G_2$ is such that $\phi(x)$ is the identity (i.e. $\phi(x)y = y$, $\forall x \in G_1, \forall y \in G_2$), then we obtain the direct product. It is easy to verify that $H_1 = \{id\}\times_\phi G_2 = \{(id,x)\,|\,x \in G_2\}$ is a normal subgroup of $G_1\times_\phi G_2$, while the subgroup $H_2 = G_1\times_\phi\{id\} = \{(x,id)\,|\,x \in G_1\}$ is not necessarily normal. Therefore Theorem 2.4 is not valid for the semi-direct product.

The composition rule for "λ dependent" elements of a reduction group is similar to the rule for a semi-direct product of the groups $\operatorname{Aut}\mathcal{R}(\Gamma)$ and $\operatorname{Aut}\mathcal{A}$. Indeed, it is easy to show that

$$(\sigma,\psi(\lambda)) = (\sigma_2,\psi_2(\lambda))\cdot(\sigma_1,\psi_1(\lambda)) = (\sigma_2\cdot\sigma_1, \psi_2(\lambda)\cdot\sigma_2(\psi_1(\lambda)))\,.$$

In this case the homomorphism $\phi: \operatorname{Aut}\mathcal{R}(\Gamma) \to \operatorname{Aut}(\operatorname{Aut}\mathcal{A})$ is the corresponding fractional linear transformation of parameter λ.

Let us consider a nontrivial example of a reduction group $\mathbb{D}_3^\lambda \cong \mathbb{D}_3$ with λ dependent automorphisms of $sl(3,\mathbb{C})$ and the corresponding infinite dimensional automorphic Lie algebra. Let $\mathbb{D}_3^\lambda$ be a group of transformations generated by

$$g_s: a(\lambda) \to Qa(\omega^{-1}\lambda)Q^{-1}, \quad g_t: a(\lambda) \to -T(\lambda)a^{tr}(\lambda^{-1})T^{-1}(\lambda), \quad a(\lambda) \in sl(3,\mathbb{C})$$

where

$$Q = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T(\lambda) = \frac{\lambda^3}{1-\lambda^6}\begin{pmatrix} 1 & \lambda^2 & \lambda^{-2} \\ \lambda^{-2} & 1 & \lambda^2 \\ \lambda^2 & \lambda^{-2} & 1 \end{pmatrix}, \quad T^{-1}(\lambda) = \begin{pmatrix} 0 & \lambda^{-1} & -\lambda \\ -\lambda & 0 & \lambda^{-1} \\ \lambda^{-1} & -\lambda & 0 \end{pmatrix}.$$

It is easy to check that $g_s^3 = id$. Also one can check that $g_t^2 = id$. Indeed, since $T(\lambda)(T^{-1}(\lambda^{-1}))^{tr} = -I$ we have

$$g_t\cdot g_t: a(\lambda) \to -T(\lambda)\left(-T(\lambda^{-1})a^{tr}(\lambda)T^{-1}(\lambda^{-1})\right)^{tr}T^{-1}(\lambda) = T(\lambda)(T^{-1}(\lambda^{-1}))^{tr}a(\lambda)(T(\lambda^{-1}))^{tr}T^{-1}(\lambda) = a(\lambda).$$

Similarly, one can check that $g_s\cdot g_t\cdot g_s\cdot g_t = id$. Thus, the group $\mathbb{D}_3^\lambda \cong \mathbb{D}_3$.

Let us describe the space of $\mathbb{D}_3^\lambda$ invariant 3×3 matrices with rational entries in λ and with simple, double and third order poles at points $\{0, \infty\}$. Matrix $a(\lambda)$ is $\mathbb{D}_3^\lambda$ invariant if and only if

$$a(\lambda) = Qa(\omega^{-1}\lambda)Q^{-1}, \qquad a(\lambda) = -T(\lambda)a^{tr}(\lambda^{-1})T^{-1}(\lambda). \tag{41}$$

Proposition 3.6 *The zero matrix is the only constant and $\mathbb{D}_3^\lambda$ invariant. If a matrix is rational in λ with poles at $\{0, \infty\}$ and $\mathbb{D}_3^\lambda$ invariant, then it can be uniquely represented as a linear combination of:*

1. *in the case of simple poles*

$$x_1(\lambda) = \mathbf{e}_{12}\lambda^{-1} - \mathbf{e}_{13}\lambda, \quad x_2(\lambda) = \mathbf{e}_{23}\lambda^{-1} - \mathbf{e}_{21}\lambda, \quad x_3(\lambda) = \mathbf{e}_{31}\lambda^{-1} - \mathbf{e}_{32}\lambda; \tag{42}$$

2. *in the case of double poles*

$$y_1(\lambda) = [x_2(\lambda), x_3(\lambda)], \quad y_2(\lambda) = [x_3(\lambda), x_1(\lambda)], \quad y_3(\lambda) = [x_1(\lambda), x_2(\lambda)], \tag{43}$$

   and $x_i(\lambda)$ listed in (42);

3. *in the case of third order poles*

$$z_1(\lambda) = [x_1(\lambda), y_1(\lambda)], \quad z_2(\lambda) = [x_2(\lambda), y_2(\lambda)], \quad z_3(\lambda) = (\lambda^3 - \lambda^{-3})I, \tag{44}$$

   and $x_i(\lambda), y_i(\lambda)$ listed in (42), (43).

Proof If matrix a is constant, then it follows from the first condition (41) that a is diagonal. The second condition means that the constant, diagonal matrix a anti-commutes with $T(\lambda)$, which is impossible if $a \neq 0$. If the matrix $a(\lambda)$ has simple poles at $\{0, \infty\}$, it can be represented as $a(\lambda) = a_0 + \lambda a_+ + \lambda^{-1}a_-$, where a_0, $a_\pm$ are constant complex matrices. From the first condition (41) it follows that

$$a(\lambda) = \begin{pmatrix} a_{11} & \lambda^{-1}a_{12} & \lambda a_{13} \\ \lambda a_{21} & a_{22} & \lambda^{-1}a_{23} \\ \lambda^{-1}a_{31} & \lambda a_{32} & a_{33} \end{pmatrix}, \qquad a_{ij} \in \mathbb{C}.$$

The second condition (41) can be rewritten as $a(\lambda)T(\lambda) + T(\lambda)a^{tr}(\lambda^{-1}) = 0$ and it is equivalent to a system of linear, homogeneous equations for constant entries a_{ij}. This system has three nontrivial solutions which can be written in the form (42). In the case of second order poles we represent $a(\lambda)$ as $a_0 + \lambda a_+ + \lambda^{-1}a_- + \lambda^2 b_+ + \lambda^{-2}b_-$. Conditions (41) yield to a system of linear equations for the constant matrices a_0, $a_\pm$, $b_\pm$, whose general solution can be written in the form $a(\lambda) = \sum_{i=1}^3 \alpha_i y_i(\lambda) + \beta_i x_i(\lambda)$, $\alpha_i, \beta_i \in \mathbb{C}$. The case of third order poles can be treated similarly. $\square$

Proposition 3.7 *1. The set*

$$\{x_i^n = x_i(\lambda)f^n, y_i^n = y_i(\lambda)f^n, h_j^n = z_j(\lambda)f^n \mid i \in \{1,2,3\}, \ j \in \{1,2\}, \ n \in \mathbb{Z}, f = \lambda^3 + \lambda^{-3}\}, \tag{45}$$

   is a basis of the automorphic Lie algebra $sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; 0, \exp(\pi i/6))$.

2. *$sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; 0, \exp(\pi i/6))$ is a direct sum of two subalgebras $sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; 0)$ and $sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; \exp(\pi i/6))$.*

3. *The subsets $\{x_i^n, y_i^n, z_j^n \mid n \geq 0\}$ and $\{x_i^n, y_i^n, z_j^n \mid n < 0\}$ of the set (45) are bases of subalgebras $sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; 0)$ and $sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; \exp(\pi i/6))$, respectively.*

4. *$sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; 0)$ is generated by $x_1(\lambda), x_2(\lambda)$ and $x_3(\lambda)$.*

Proof The proof of the first statement of the Proposition is similar to the proofs of Proposition 3.1 and Proposition 3.3. The proof of the rest follows from the commutation relations for the basis elements of the algebra

$$[x_i^n, x_j^m] = \epsilon_{ijk} y_k^{n+m}, \ [y_i^n, y_j^m] = -\epsilon_{ijk}(x_k^{n+m+1} - y_i^{n+m} - y_j^{n+m}), \ [x_i^n, y_j^m] = -2\epsilon_{ijk} x_i^{n+m}, \quad i \neq j, \quad (46)$$

$$[x_1^n, y_1^m] = h_1^{n+m}, \quad [x_2^n, y_2^m] = h_2^{n+m}, \quad [x_3^n, y_3^m] = -h_1^{n+m} - h_2^{n+m}, \quad (47)$$

$$[h_2^n, h_1^m] = \sum_{k=1}^{3}(x_k^{n+m+1} - 2y_k^{n+m}), \quad [h_i^n, x_i^m] = 2x_i^{n+m+1}, \quad i = 1, 2, \quad (48)$$

$$\begin{aligned} [h_1^n, y_1^m] &= -h_1^{n+m} - 2h_2^{n+m} + 2(x_2^{n+m} + x_3^{n+m} - y_1^{n+m+1}), \\ [h_2^n, y_2^m] &= 2h_1^{n+m} + h_2^{n+m} + 2(x_1^{n+m} + x_3^{n+m} - y_2^{n+m+1}), \end{aligned} \quad (49)$$

$$[h_i^n, x_j^m] = -x_j^{n+m+1} + y_i^{n+m} - |\epsilon_{ijk}| y_k^{n+m}, \quad [h_i^n, y_j^m] = -y_j^{n+m+1} - 2x_i^{n+m} - \epsilon_{ijk} h_i^{n+m}, \quad i \neq j. \quad (50)$$

Elements with non-negative upper index form a closed $\mathbb{D}_3^\lambda$–automorphic subalgebra and they have poles at points $\{0, \infty\}$. This subalgebra is generated by $x_i^0 = x_i(\lambda)$. Indeed, y_i^0 can be found from (46), h_i^0 from (47), x_i^1, y_i^1 from (49), etc.

Elements with negative upper index also form a closed subalgebra, they have poles at the points of the orbit $\{\exp\left(\frac{(2n+1)\pi i}{6}\right) \,|\, n = 1, ..., 6\}$ and are regular elsewhere. $\square$

Algebra $sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; 0)$ has been discovered in [11], but its automorphic nature and the reduction group was not known until now. It is not difficult to show that it does not exist any λ independent reduction group which corresponds to $sl_{\mathbb{D}_3^\lambda}(3, \mathbb{C}; 0)$.

A   Appendix. Finite groups of fractional-linear transformations, their orbits and primitive automorphic functions

The group $\mathbb{Z}_N$

The group $\mathbb{Z}_N$ can be represented by the following transformations

$$\sigma_n(\lambda) = \Omega^n \lambda, \qquad \Omega = \exp\left(\frac{2\pi i}{N}\right), \qquad n = 0, 1, \ldots, N-1. \quad (51)$$

It has two degenerated orbits $\mathbb{Z}_N(0) = \{0\}$, $\mathbb{Z}_N(\infty) = \{\infty\}$ corresponding to two fixed points of order N and a generic orbit $\mathbb{Z}_N(\gamma) = \{\gamma, \Omega\gamma, \Omega^2\gamma, \ldots, \Omega^{N-1}\gamma\}$, $\gamma \notin \{0, \infty\}$. A primitive automorphic function, corresponding to the orbits $\mathbb{Z}_N(0)$, $\mathbb{Z}_N(\infty)$ is

$$f_{\mathbb{Z}_N}(\lambda, \infty, 0) = \lambda^N.$$

It follows from (4)–(6) that for $\gamma_1 \neq \infty$ and $\gamma_2 \notin \mathbb{Z}_N(\gamma_1)$

$$f_{\mathbb{Z}_N}(\lambda, \gamma_1, \gamma_2) = \frac{\lambda^N - \gamma_2^N}{\lambda^N - \gamma_1^N}.$$

The dihedral group $\mathbb{D}_N$

The group $\mathbb{D}_N$ has order $2N$ and can be generated by the following transformations

$$\sigma_s(\lambda) = \Omega\lambda, \qquad \sigma_t(\lambda) = \frac{1}{\lambda}, \qquad \Omega = \exp\left(\frac{2\pi i}{N}\right). \quad (52)$$

Transformations σ_s, σ_b satisfy the relations $\sigma_s^N = \sigma_t^2 = (\sigma_s \sigma_t)^2 = id$ and

$$\mathbb{D}_N = \{\sigma_s^n, \sigma_s^n \sigma_t \mid n = 0, \ldots, N - 1\}\,.$$

For $N \geq 3$ the group $\mathbb{D}_N$ is non-commutative, the case $N = 2$ is special, in this case $\mathbb{D}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and it is commutative. F. Klein calls it the quadratic group (some authors call $\mathbb{D}_2$ the Klein group).

The group $\mathbb{D}_N$ has three degenerated orbits and one generic orbit. The structure of the orbits is different for odd and even N. For odd N we have:

$$\begin{align}
\mathbb{D}_N(0) &= \{0, \infty\}\,, \tag{53}\\
\mathbb{D}_N(1) &= \{\, 1, \Omega, \ldots, \Omega^{N-1}\}\,, \tag{54}\\
\mathbb{D}_N(-1) &= \{\, -1, -\Omega, \ldots, -\Omega^{N-1}\}\,, \tag{55}\\
\mathbb{D}_N(\gamma) &= \{\, \gamma, \Omega\gamma, \ldots, \Omega^{N-1}\gamma, \gamma^{-1}, \Omega\gamma^{-1}, \ldots, \Omega^{N-1}\gamma^{-1}\}\,. \tag{56}
\end{align}$$

For even N orbits $\mathbb{D}_N(1)$ and $\mathbb{D}_N(-1)$ coincide and instead of (55) we have the orbit

$$\mathbb{D}_N(i) = \{\, i, i\Omega, \ldots, i\Omega^{N-1}\}\,. \tag{57}$$

The orbit (53) consists of fixed points of order N. Orbits (54), (55), (57) consist of fixed points of order 2 (they correspond to the vertices of the dihedron or to the middles of the edges, i.e. vertices of the dual dihedron). A primitive automorphic function, corresponding to the orbits $\mathbb{D}_N(0)$, $\mathbb{D}_N(1)$ is

$$f_{\mathbb{D}_N}(\lambda, 0, 1) = \lambda^N + \lambda^{-N} - 2\,.$$

The tetrahedral group $\mathbb{T}$

The group of a tetrahedron $\mathbb{T}$ has order 12 and can be generated by two transformations

$$\sigma_s(\lambda) = -\lambda\,, \quad \sigma_t(\lambda) = \frac{\lambda + i}{\lambda - i}\,.$$

It is easy to check that $\sigma_s^2 = \sigma_t^3 = (\sigma_s \sigma_t)^3 = id$ and

$$\mathbb{T} = \{\sigma_t^n, \sigma_t^n \sigma_s \sigma_t^m \mid n, m = 0, 1, 2\}\,.$$

The group $\mathbb{T}$ has four distinct orbits. The orbit corresponding to a generic point γ is a set of 12 points

$$\mathbb{T}(\gamma) = \{\pm\gamma, \pm\gamma^{-1}, \pm i\frac{\gamma + 1}{\gamma - 1}, \pm i\frac{\gamma - 1}{\gamma + 1}, \pm\frac{\gamma + i}{\gamma - i}, \pm\frac{\gamma - i}{\gamma + i}\}\,,$$

Transformation σ_a has two fixed points of order two, namely $\{0, \infty\}$, the corresponding orbit consists of six points, which correspond to middle of the edges of the tetrahedron

$$\mathbb{T}(0) = \{0, \infty, \pm 1, \pm i\}\,. \tag{58}$$

There are two orbits with fixed points of order 3. They correspond to the vertices of the tetrahedron and the dual tetrahedron. Fixed points of the transformation σ_t can be used as seeds for these orbits. It follows from $\gamma = \sigma_t(\gamma)$ that the fixed points are $\gamma_1 = (1+i)/(1+\sqrt{3}) = \omega + i\bar{\omega}$, $\gamma_2 = (1+i)/(1-\sqrt{3}) = i\omega + \bar{\omega}$, where $\omega = \exp(2\pi i/3)$ and therefore we have two orbits:

$$\mathbb{T}(\gamma_1) = \{\pm(\omega + i\bar{\omega}), \pm(\omega - i\bar{\omega})\}\,, \qquad \mathbb{T}(\gamma_2) = \{\pm i(\omega + i\bar{\omega}), \pm i(\omega - i\bar{\omega})\}\,. \tag{59}$$

Points of the orbits $\mathbb{T}(\gamma_1)$ and $\mathbb{T}(\gamma_2)$ are roots of the equations $\lambda^4 + 2(\omega + \bar{\omega})\lambda^2 + 1 = 0$ and $\lambda^4 - 2(\omega + \bar{\omega})\lambda^2 + 1 = 0$, respectively. A primitive automorphic function, corresponding to orbits $\mathbb{T}(\gamma_1)$, $\mathbb{T}(\gamma_2)$ is

$$f_{\mathbb{T}}(\lambda, \gamma_1, \gamma_2) = \left(\frac{\lambda^4 + 2(\omega + \bar{\omega})\lambda^2 + 1}{\lambda^4 - 2(\omega + \bar{\omega})\lambda^2 + 1}\right)^3\,.$$

It follows from (5) that

$$f_{\mathbb{T}}(\lambda, \gamma_1, 0) = f_{\mathbb{T}}(\lambda, \gamma_1, \gamma_2) - 1 = 12(\omega + \bar{\omega})\frac{\lambda^2(\lambda^4 - 1)^2}{(\lambda^4 - 2(\omega + \bar{\omega})\lambda^2 + 1)^3}\,.$$

The octahedral group $\mathbb{O}$

The group of an octahedron $\mathbb{O}$ has order 24 and can be generated by two transformations

$$\sigma_s(\lambda) = i\lambda, \quad \sigma_t(\lambda) = \frac{\lambda + 1}{\lambda - 1}. \tag{60}$$

It is easy to check that $\sigma_s^4 = \sigma_t^2 = (\sigma_s\sigma_t)^3 = id$ and

$$\mathbb{O} = \{\sigma_s^n, \sigma_s^n\sigma_t\sigma_s^m, \sigma_s^n\sigma_t\sigma_s^2\sigma_t \mid n, m = 0, 1, 2, 3\}.$$

The group $\mathbb{O}$ has also four distinct orbits corresponding to

i the vertices of the octahedron (a fixed point of order 4 of the transformation σ_s belongs to this orbit), therefore

$$\mathbb{O}(0) = \mathbb{T}(0);$$

ii the centres of the triangular faces (i.e. vertices of the cube - the dual to the octahedron). the point γ_1, a fixed point of σ_t, belongs to this orbit, therefore

$$\mathbb{O}(\gamma_1) = \mathbb{T}(\gamma_1) \bigcup \mathbb{T}(\gamma_2);$$

iii the middles of the edges of the octahedron

$$\mathbb{O}(\delta) = \{\pm\delta, \ \pm\bar{\delta}, \ i^n(1 + \delta + \bar{\delta}), \ i^n(1 - \delta - \bar{\delta}) \mid n = 0, 1, 2, 3\}$$

where $\delta = \exp(\pi i/4)$ is one of the points on the middle of an edge of the octahedron, for example a fixed point of the transformation $\lambda \to i/\lambda$, which belongs to the group generated by σ_s, σ_t (60));

iv the orbit, corresponding to a generic point γ (i.e. γ does not belong to the above listed orbits) is a set of 24 points

$$\mathbb{O}(\gamma) = \{i^k\gamma, i^k\gamma^{-1}, i^k\frac{\gamma + 1}{\gamma - 1}, i^k\frac{\gamma - 1}{\gamma + 1}, i^k\frac{\gamma + i}{\gamma - i}, i^k\frac{\gamma - i}{\gamma + i}\}, \qquad k \in \{0, 1, 2, 3\}.$$

A primitive automorphic function, corresponding to orbits $\mathbb{O}(0), \mathbb{O}(\gamma_1)$ is

$$f_{\mathbb{O}}(\lambda, 0, \gamma_1) = \frac{(\lambda^4 - 2(\omega + \bar{\omega})\lambda^2 + 1)^3(\lambda^4 + 2(\omega + \bar{\omega})\lambda^2 + 1)^3}{\lambda^4(\lambda^4 - 1)^4} = \frac{(\lambda^8 + 14\lambda^4 + 1)^3}{\lambda^4(\lambda^4 - 1)^4}.$$

The icosahedral group $\mathbb{I}$

The group of the icosahedron $\mathbb{I}$ has order 60 and can be generated by two transformations

$$\sigma_s(\lambda) = \varepsilon\lambda, \quad \sigma_t(\lambda) = \frac{(\varepsilon^2 + \varepsilon^3)\lambda + 1}{\lambda - \varepsilon^2 - \varepsilon^3}, \quad \varepsilon = \exp\left(\frac{2\pi i}{5}\right). \tag{61}$$

Its generators satisfy the relations $\sigma_s^5 = \sigma_t^2 = (\sigma_s\sigma_t)^3 = id$ and

$$\mathbb{I} = \{\sigma_s^n, \ \sigma_s^n\sigma_t\sigma_s^m, \ \sigma_s^n\sigma_t\sigma_s^2\sigma_t\sigma_s^m, \ \sigma_s^n\sigma_t\sigma_s^2\sigma_t\sigma_s^3\sigma_t \mid n, m = 0, 1, 2, 3, 4\}.$$

The group $\mathbb{I}$ has also four distinct orbits corresponding to

i the vertices of the icosahedron (fixed points of order 5 of the transformation σ_s belong to this orbit)

$$\mathbb{I}(0) = \{0, \infty, \varepsilon^{k+1} + \varepsilon^{k-1}, \varepsilon^{k+2} + \varepsilon^{k-2} \mid k = 0, 1, 2, 3, 4\}.$$

The finite points of this orbit are all solutions of the equation $\lambda(\lambda^{10} + 11\lambda^5 - 1) = 0$.

ii the centres of the triangular faces (i.e. vertices of dodecahedron - the dual to the icosahedron). The transformation

$$\sigma_s^2 \sigma_t \sigma_s^2(\lambda) = \frac{(1 + \bar{\varepsilon})\lambda + 1}{\lambda - 1 - \varepsilon}$$

is of order 3 and it has fixed points

$$\gamma_1 = \frac{3 + \sqrt{5} + \sqrt{6(5 + \sqrt{5})}}{4} = 1 - \omega\varepsilon - \bar{\omega}\bar{\varepsilon}, \quad \gamma_2 = \frac{3 + \sqrt{5} - \sqrt{6(5 + \sqrt{5})}}{4} = 1 - \bar{\omega}\varepsilon - \omega\bar{\varepsilon}$$

(here $\omega = \exp(2\pi i/3)$). The corresponding orbit $\mathbb{I}(\gamma_1)$ consists of 20 points, these points are solutions of the equation [2]

$$\lambda^{20} - 228\lambda^{15} + 494\lambda^{10} + 228\lambda^5 + 1 = 0.$$

iii the middles of the edges of the icosahedron correspond to the orbit $\mathbb{I}(i)$. The point i is a fixed point of transformation $\sigma_s^2 \sigma_t \sigma_s^3 \sigma_t \sigma_s^2 \sigma_t(\lambda) = -1/\lambda$. Points of this orbit are solutions of the equation

$$\lambda^{30} + 522\lambda^{25} - 10005\lambda^{20} - 10005\lambda^{10} - 522\lambda^5 + 1 = 0.$$

iv the orbit, corresponding to a generic point γ (i.e. γ does not belong to the above listed orbits) is a set of 60 points

$$\mathbb{I}(\gamma) = \left\{\varepsilon^n \gamma, \frac{\varepsilon^n}{\gamma}, \ \varepsilon^n \frac{(\varepsilon^3 + \varepsilon^2)\varepsilon^m \gamma + 1}{\varepsilon^m \gamma - \varepsilon^3 - \varepsilon^2}, \ -\varepsilon^n \frac{\varepsilon^m \gamma - \varepsilon^3 - \varepsilon^2}{(\varepsilon^3 + \varepsilon^2)\varepsilon^m \gamma + 1} \ \middle| \ n, m = 0, 1, 2, 3, 4\right\}.$$

A primitive automorphic function, corresponding to orbits $\mathbb{I}(0)$, $\mathbb{I}(\gamma_1)$ is

$$f_{\mathbb{I}}(\lambda, 0, \gamma_1) = \frac{(\lambda^{20} - 228\lambda^{15} + 494\lambda^{10} + 228\lambda^5 + 1)^3}{\lambda^5(\lambda^{10} + 11\lambda^5 - 1)^5}.$$

It is easy to check that

$$f_{\mathbb{I}}(\lambda, 0, i) = f_{\mathbb{I}}(\lambda, 0, \gamma_1) - f_{\mathbb{I}}(i, 0, \gamma_1) = \frac{(\lambda^{30} + 522\lambda^{25} - 10005\lambda^{20} - 10005\lambda^{10} - 522\lambda^5 + 1)^2}{\lambda^5(\lambda^{10} + 11\lambda^5 - 1)^5}.$$

Acknowledgement

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